

Note on recursion relations for the Q -cut representation

Bo Feng,^{a,b} Song He,^{c,d} Rijun Huang^{1a} and Ming-xing Luo^a

^a*Zhejiang Institute of Modern Physics, Department of Physics, Zhejiang University,
No.38, Zheda Road, Hangzhou, 310027, P.R. China.*

^b*Center of Mathematical Science, Zhejiang University,
No.38, Zheda Road, Hangzhou, 310027, P.R. China.*

^c*CAS Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences,
No. 55, ZhongGuanCun East Street, Beijing 100190, P.R.China.*

^d*School of Physical Sciences, University of Chinese Academy of Sciences,
No. 19A, Yuquan Road, Beijing 100049, P.R.China.*

E-mail: fengbo@zju.edu.cn, songhe@itp.ac.cn, huang@nbi.dk, mingxingluo@zju.edu.cn

ABSTRACT: In this note, we study the Q -cut representation by combining it with BCFW deformation. As a consequence, the one-loop integrand is expressed in terms of a recursion relation, *i.e.*, n -point one-loop integrand is constructed using tree-level amplitudes and m -point one-loop integrands with $m \leq n - 1$. By giving explicit examples, we show that the integrand from the recursion relation is equivalent to that from Feynman diagrams or the original Q -cut construction, up to scale free terms.

KEYWORDS: Scattering Amplitude, Loop Integrand

¹The correspondence author.

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1 Introduction

In a recent work, a new representation of the perturbative S -matrix, known as \mathcal{Q} -cut representation, was proposed [1]. It allows one to write the integrand of loop amplitude as summation of products of lower-point tree-level amplitudes with deformed loop momenta. For generic n -point one-loop integrand with all massless external legs, the new representation takes the form,

$$\mathcal{I}_n^{\mathcal{Q}}(\ell) = \sum_{P_L} \sum_{h_1, h_2} \mathcal{A}_L(\cdots, \widehat{\ell}_R^{h_1}, -\widehat{\ell}_L^{h_2}) \frac{1}{\ell^2(-2\ell \cdot P_L + P_L^2)} \mathcal{A}(\widehat{\ell}_L^{\bar{h}_2}, -\widehat{\ell}_R^{\bar{h}_1}, \cdots), \quad (1.1)$$

where $\widehat{\ell} = \alpha_L(\ell + \eta)$, $\widehat{\ell}_R \equiv \widehat{\ell}_L - P_L$ with $\alpha_L = P_L^2/(2\ell \cdot P_L) \neq 0$, $\eta^2 = \ell^2$. As will be reviewed shortly, two deformations have been applied to the loop momentum ℓ : firstly the dimensional deformation $\ell \rightarrow \ell + \eta$ with η in extra dimensions, and secondly the scale deformation $\ell \rightarrow \alpha\ell$. The details of the one-loop \mathcal{Q} -cut construction was further clarified in [2], and generalizations to two loops or more was also illustrated in [1]. The \mathcal{Q} -cut representation circumvented two difficulties in the attempt for recursive construction of loop integrand: canonical definition of loop momentum and the singularities in the forward limit (which will be referred to as forward singularities). On the other hand, the integration over loop momentum with such integrand still requires more systematic investigations.

The \mathcal{Q} -cut representation was partly inspired by the work [3] and finds direct application in the study of writing one-loop amplitudes based on the Riemann sphere [4–6]¹, and very recently in an extension

¹In the scattering equation formalism [7–11], loop integrands for super-gravity and super-Yang-Mills amplitude has formerly been proposed [3], since in these theories there is no forward singularity.

to two-loop supersymmetric amplitudes from Riemann sphere [12]. Another work also reports similar one-loop integrand expansion while investigating elliptic scattering equations at one-loop level [13], based on an earlier work on the Λ scattering equation [14]. The idea in the \mathcal{Q} -cut construction also inspires some thoughts in the other approach of constructing one-loop amplitude [15], as well as the construction of two-loop planar integrand of cubic scalar theory [16]. These works have shown the universality and importance of \mathcal{Q} -cut representation for loop integrands in general.

After the discovery of Britto-Cachazo-Feng-Witten(BCFW) recursion relations for tree-level amplitudes [17, 18], it is very natural to ask if one can construct loop integrands in a similar, recursive way. The key for the progress lies in expressing planar loop integrands from forward limits of tree amplitudes [19–21], which has been very successful for cases without forward singularities, such as super-Yang-Mills at one loop and planar $\mathcal{N} = 4$ SYM to all loops [20]. However, for general theories the afore-mentioned difficulties have only been resolved in the \mathcal{Q} -cut construction. These works have indicated clearly that for generic loop integrands, BCFW deformation has to be applied with extra care, especially due to the presence of forward singularities. In the \mathcal{Q} -cut construction, the dimensional deformation transforms one-loop integrand into tree diagrams, while the scale deformation has avoided the forward singularities by excluding the tree diagrams that corresponding to one-loop tadpole and massless bubble contributions, which should not be presented in the final amplitude.

Both recursion relations and \mathcal{Q} -cut approach to the construction of loop integrands in general theories are promising but with some unsatisfying features: the \mathcal{Q} -cut representation has non-standard propagators, while it is not clear how to remove forward singularities in general in recursion relations. Thus it is natural to see if by combining the two methods to make further progress. In this note, we will initiate the study along this direction. We would like to see if there is another way to deal with forward singularities and how much can we learn about the structure of one-loop integrands from both recursion and \mathcal{Q} -cut viewpoints.

This paper is structured as follows. In §2, we illustrate the application of BCFW deformation in the \mathcal{Q} -cut construction, and present a recursive formula for one-loop integrand. In §3, we explain the details of the recursive formula by three examples, and confirm the validity of the results by comparing with results from one-loop Feynman diagrams and those from the \mathcal{Q} -cut construction. We conclude in §4.

2 The derivation of recursion relation

Let us first recall the original derivation of \mathcal{Q} -cut representation in [1]. After imposing the dimensional deformation $\ell \rightarrow \ell + \eta$ as well as the shift $\ell \rightarrow \ell + P$ for loop momentum, the n -point one-loop integrand $\mathcal{I}^{\mathcal{Q}}(\ell)$ becomes essentially the $(n+2)$ -point tree-level amplitude $\mathcal{T}(\ell)$, on the condition $\ell^2 = 0$. Then by scale deformation $\ell \rightarrow \alpha\ell$, and by removing diagrams that contribute to one-loop tadpoles and massless bubbles appropriately, one gets the one-loop integrand. Since BCFW recursion has been applied to the computation of ordinary tree-level amplitudes, this naturally motivates us to consider the possibility of constructing the $(n+2)$ -point tree-level amplitude $\mathcal{T}(\ell)$ using the recursion. Here we present a derivation of the recursive

representation for one-loop integrand following the afore-mentioned motivation. The derivation will take three steps, as follows.

2.1 Step one: dimensional deformation

Just like the original \mathcal{Q} -cut construction [1], the first step of the derivation is to reformulate one-loop integrand in terms of tree-level amplitudes. We take the same dimensional deformation $\ell \rightarrow \ell + \eta$ as in [1] and also the loop momentum shifting, to arrive at

$$\mathcal{A}^{1\text{-loop}} = \int d^D \ell \mathcal{I}^{\mathcal{Q}}(\ell) \quad , \quad \mathcal{I}^{\mathcal{Q}}(\ell) = \frac{1}{\ell^2} \mathcal{T}^{\mathcal{Q}}(\ell) . \quad (2.1)$$

Some explanations are in order for (2.1). Firstly, from the dimensional deformation, it is known that $\mathcal{T}^{\mathcal{Q}}$ is given by those Feynman diagrams with n external legs and two extra legs by cutting an internal propagator. Thus $\mathcal{T}^{\mathcal{Q}}$ is defined on the condition $\ell^2 = 0$, which says that all ℓ in $\mathcal{T}^{\mathcal{Q}}$ should be understood as the null momentum in higher dimension. Furthermore, $\mathcal{T}^{\mathcal{Q}}$ is not exactly the full $(n+2)$ -point tree-level amplitude, since in order to reconstruct the one-loop integrand, some diagrams should be excluded. Such tree-level diagrams correspond to one-loop tadpole and massless bubble diagrams with single cuts. From Feynman diagrams one can inspect that, a tadpole after single cut will produce tree diagrams with $\ell, -\ell$ attaching to the same vertex², while massless bubble diagram with the massless leg p_i after single cut will produce tree diagrams with ℓ, p_i (or $-\ell, p_i$) attaching to the same three-point vertex, and then meeting $-\ell$ (or ℓ) in the neighboring vertex. The above scenery would help us to exclude corresponding tree diagrams in the following steps.

Next let us take a look at the contributing tree diagrams to $\mathcal{T}^{\mathcal{Q}}$. If the theory under consideration is not color-ordered, we shall consider the full $(n+2)$ -point on-shell tree-level Feynman diagrams after removing those corresponding to the one-loop tadpole and massless bubbles. While if it is color-ordered, the $\mathcal{T}^{\mathcal{Q}}$ gets contribution from n different color-ordered tree diagrams, each by breaking an internal line of the n propagators. Since there are n different color orderings, we can calculate each one independently, for example, using different methods (such as Feynman diagrams or BCFW recursion relations) or different deformations in BCFW recursion relations.

A final remark says that, the loop momentum shifting in expression (2.1) makes a canonical definition of loop momentum, such that the integrand is irrelevant to the labeling of ℓ for internal propagators.

2.2 Step two: BCFW deformation

Now let us turn to $\mathcal{T}^{\mathcal{Q}}$, and our aim is to determine it by BCFW deformation. Since it is effectively tree-level amplitude but with forward singularity removed, the analysis on the large z behavior would be the same and the computation should be straightforward. Let us, for generality, take two arbitrary momenta p_i, p_j (but not $\ell, -\ell$) and perform the standard BCFW deformation

$$\hat{p}_i = p_i + zq \quad , \quad \hat{p}_j = p_j - zq \quad \text{with} \quad q^2 = q \cdot p_i = q \cdot p_j = 0 . \quad (2.2)$$

²Here $\ell, -\ell$ denotes two legs by breaking an internal line.

Such deformation can be realized when the dimension $D \geq 4$. In this case, $\mathcal{T}^{\mathcal{Q}}$ becomes an analytic function of external momenta p_i 's, loop momentum ℓ and a complex variable z . As usual, we can consider the contour integration

$$\oint_{\Gamma} \frac{dz}{z} \mathcal{T}^{\mathcal{Q}}(z) , \quad (2.3)$$

where the contour Γ is a very large circle. This integration leads to

$$\mathcal{T}^{\mathcal{Q}}(z=0) = \mathcal{B} + \sum_{z=z_{\gamma}} \frac{\mathcal{T}^{\mathcal{Q}}}{z} , \quad (2.4)$$

where the sum is over all finite pole z_{γ} 's of $\mathcal{T}^{\mathcal{Q}}$, and \mathcal{B} is possible boundary contribution. It is well-known for tree-level amplitudes that for Yang-Mills and gravity theories, the BCFW deformation can be chosen such that the boundary contribution vanishes. While for some other theories, the boundary contribution would appear and require more careful analysis [22–30]. Here we shall assume $\mathcal{B} = 0$ for simplicity (but the similar consideration can be generalized to the case with non-zero boundary contributions). Thus the only information we need for computing $\mathcal{T}^{\mathcal{Q}}$ by means of expression (2.4) is the pole structure of function $\mathcal{T}^{\mathcal{Q}}(z)$.

The BCFW deformation splits a tree amplitude into two parts, with the shifted momenta \hat{p}_i, \hat{p}_j locating in each part. Assuming $\hat{K}_{\gamma} \equiv \hat{p}_i + P_{\gamma}$ is the sum of all momenta in the part containing \hat{p}_i , and $K_{\gamma} \equiv p_i + P_{\gamma}$. From $\hat{K}_{\gamma}^2 = 0$ we get $z_{\gamma} = -K_{\gamma}^2/(2q \cdot K_{\gamma})$. Now let us consider the two extra legs $\ell, -\ell$. If they are in the same part, K_{γ} will have no dependence on ℓ , thus also the pole z_{γ} . We shall denote the corresponding contribution as $\mathcal{R}_A^{\mathcal{Q}}$. While if $\ell, -\ell$ are separated in two parts, K_{γ} as well as z_{γ} would depend on ℓ . We shall denote the corresponding contribution as $\mathcal{R}_B^{\mathcal{Q}}$. So we have

$$\mathcal{T}^{\mathcal{Q}} = \mathcal{R}_A^{\mathcal{Q}} + \mathcal{R}_B^{\mathcal{Q}} . \quad (2.5)$$

For the contribution $\mathcal{R}_A^{\mathcal{Q}}$, we can further organize it into two parts,

$$\mathcal{R}_A^{\mathcal{Q}} = \mathcal{R}_{A,1}^{\mathcal{Q}} + \mathcal{R}_{A,2}^{\mathcal{Q}} . \quad (2.6)$$

$\mathcal{R}_{A,1}^{\mathcal{Q}}$ denotes the contribution where legs $\ell, -\ell$ are in the part containing \hat{p}_j , while $\mathcal{R}_{A,2}^{\mathcal{Q}}$ denotes the contribution where legs $\ell, -\ell$ are in the part containing \hat{p}_i . Explicitly, we have

$$\mathcal{R}_{A,1}^{\mathcal{Q}} = \sum_{h,\gamma} A(\hat{p}_i(z_{\gamma}), \{\gamma\}, -\hat{K}_{\gamma}^h(z_{\gamma})) \frac{1}{K_{\gamma}^2} \mathcal{T}(\hat{K}_{\gamma}^{-h}(z_{\gamma}), \hat{p}_j(z_{\gamma}), \{\beta\}, \ell, -\ell) , \quad (2.7)$$

where

$$z_{\gamma} = -\frac{(P_{\gamma} + p_i)^2}{2q \cdot P_{\gamma}} , \quad \hat{K}_{\gamma}(z_{\gamma}) = P_{\gamma} + p_i + z_{\gamma} q ,$$

as well as $\hat{p}_i(z_{\gamma}) = p_i + z_{\gamma} q$, $\hat{p}_j(z_{\gamma}) = p_j - z_{\gamma} q$, and $\{\gamma\} \cup \{\beta\} = \{1, 2, \dots, n\}/\{i, j\}$. Similarly,

$$\mathcal{R}_{A,2}^{\mathcal{Q}} = \sum_{h,\beta} \mathcal{T}(\ell, -\ell, \{\gamma\}, \hat{p}_i(z_{\beta}), -\hat{K}_{\beta}^h(z_{\beta})) \frac{1}{K_{\beta}^2} A(\hat{K}_{\beta}^{-h}(z_{\beta}), \{\beta\}, \hat{p}_j(z_{\beta})) , \quad (2.8)$$

where

$$z_\beta = \frac{(P_\beta + p_j)^2}{2q \cdot P_\beta} \quad , \quad \widehat{K}_\beta(z_\beta) = -(P_\beta + p_j - z_\beta q) \quad .$$

Note that the sum is over all possible splitting of $(n-2)$ legs $\{1, 2, \dots, n\}/\{i, j\}$ and helicities. Also note that inside the bracket $A(\bullet), \mathcal{T}(\bullet)$ we have explicitly labeled all the legs in each part but not the ordering of legs. The color-ordering of legs should be understood with respect to their corresponding theories.

Now let us take a more careful look on expressions (2.7) and (2.8). Firstly, the \mathcal{T} part in $\mathcal{R}_{A,1}^{\mathcal{Q}}, \mathcal{R}_{A,2}^{\mathcal{Q}}$ will be lower-point on-shell tree diagrams after excluding those corresponding to tadpole and bubble diagrams. This means that when dressing with $\frac{1}{\ell^2}$, they would become lower-point one-loop integrand, which can be obtained by any legitimate methods, such as the original \mathcal{Q} -cut construction or Feynman diagram method with partial fraction identity. One important implication is that the forward singularities in the type \mathcal{R}_A have been automatically removed after using the well-defined one-loop integrands of lower points. Secondly, for $\mathcal{R}_{A,1}^{\mathcal{Q}}$, the number of legs in set $\{\gamma\}$ must be at least one, in order for the amplitude to be non-vanishing. Naively, the number of legs in set $\{\beta\}$ could also be zero. However, when it is so, the tree diagrams of \mathcal{T} are exactly those corresponding to tadpole and massless bubbles, which need to be excluded. So $\{\beta\}$ could not be empty set. Similarly for $\mathcal{R}_{A,2}^{\mathcal{Q}}$, the number of legs in sets $\{\gamma\}, \{\beta\}$ should at least be one.

Now let us analyze the contribution $\mathcal{R}_B^{\mathcal{Q}}$. We can also organize it into two parts,

$$\mathcal{R}_B^{\mathcal{Q}} = \mathcal{R}_{B,1}^{\mathcal{Q}} + \mathcal{R}_{B,2}^{\mathcal{Q}} \quad . \quad (2.9)$$

$\mathcal{R}_{B,1}^{\mathcal{Q}}$ denotes the contribution where leg ℓ is in the part containing \widehat{p}_i , while $\mathcal{R}_{B,2}^{\mathcal{Q}}$ denotes the contribution where leg ℓ is in the part containing \widehat{p}_j , explicitly as

$$\mathcal{R}_{B,1}^{\mathcal{Q}} = \sum_{h,\gamma} \mathcal{T}(\ell, \widehat{p}_i(z_\gamma), \{\gamma\}, -\widehat{K}_\gamma^h(z_\gamma)) \frac{1}{K_\gamma^2} \mathcal{T}(\widehat{K}_\gamma^{-h}(z_\gamma), \widehat{p}_j(z_\gamma), \{\beta\}, -\ell) \quad , \quad (2.10)$$

where

$$z_\gamma = -\frac{(P_\gamma + p_i + \ell)^2}{2q \cdot (P_\gamma + \ell)} \quad , \quad \widehat{K}_\gamma(z_\gamma) = P_\gamma + p_i + \ell + z_\gamma q \quad ,$$

and $\{\gamma\} \cup \{\beta\} = \{1, 2, \dots, n\}/\{i, j\}$. While

$$\mathcal{R}_{B,2}^{\mathcal{Q}} = \sum_{h,\gamma} \mathcal{T}(-\ell, \widehat{p}_i(z_\gamma), \{\gamma\}, -\widehat{K}_\gamma^h(z_\gamma)) \frac{1}{K_\gamma^2} \mathcal{T}(\widehat{K}_\gamma^{-h}(z_\gamma), \widehat{p}_j(z_\gamma), \{\beta\}, \ell) \quad , \quad (2.11)$$

where

$$z_\gamma = -\frac{(P_\gamma + p_i - \ell)^2}{2q \cdot (P_\gamma - \ell)} \quad , \quad \widehat{K}_\gamma(z_\gamma) = P_\gamma + p_i - \ell + z_\gamma q \quad .$$

Some discussions are in order for expressions (2.10) and (2.11). Notice that we have used \mathcal{T} instead of tree-level amplitude A , since in this stage potential contributions coming from corresponding to tadpole and

bubble diagrams in $\mathcal{R}_{B,1}^Q, \mathcal{R}_{B,2}^Q$ should be excluded. Recalling our discussion on the excluded diagrams in the previous subsection, we can conclude that, since $\ell, -\ell$ are separated into two parts, there could not be diagrams corresponding to one-loop tadpoles, while diagrams corresponding to massless bubbles³ do exist in $\mathcal{R}_{B,1}^Q$ and $\mathcal{R}_{B,2}^Q$ when the set $\{\gamma\}$ or $\{\beta\}$ is empty. In other words, forward singularities corresponding to tadpoles have been avoided in type \mathcal{R}_B . Combining the discussions for type \mathcal{R}_A , we see that we can remove forward singularities corresponding to tadpoles without using scale deformation as is done in the \mathcal{Q} -cut construction. However, forward singularities that corresponding to massless bubbles are more difficult to deal with and we will organize $\mathcal{R}_{B,1}^Q$ into three contributions

$$\mathcal{R}_{B,1}^Q = \mathcal{R}'_{B,1} + \mathcal{R}''_{B,1} + \mathcal{R}'''_{B,1} . \quad (2.12)$$

$\mathcal{R}'_{B,1}$ denotes the contribution of the case when both $\{\gamma\}$ and $\{\beta\}$ are not empty, so forward singularities corresponding to massless bubbles will not appear and there will be no excluded diagrams. Thus the \mathcal{T} is exactly the tree amplitude and we have

$$\mathcal{R}'_{B,1} = \sum_{\gamma, h}^{1 \leq |\gamma| \leq n-3} A(\ell, \hat{p}_i(z_\gamma), \{\gamma\}, -\hat{K}_\gamma^h(z_\gamma)) \frac{1}{K_\gamma^2} A(\hat{K}_\gamma^{-h}(z_\gamma), \hat{p}_j(z_\gamma), \{\beta\}, -\ell) , \quad (2.13)$$

where the sum is over all helicities and possible splitting of external legs with the length of set $\{\gamma\}$ satisfying $1 \leq |\gamma| \leq n-3$. This is to ensure that there is at least one leg in set $\{\gamma\}, \{\beta\}$.

$\mathcal{R}''_{B,1}$ denotes the special case when set $\{\gamma\} = \emptyset$. In this case, $\mathcal{T}(\ell, \hat{p}_i, \{\gamma\}, -\hat{K}_\gamma)$ becomes a three-point amplitude, and we get explicitly

$$\mathcal{R}''_{B,1} = \sum_h A(\ell, \hat{p}_i(z_\gamma), -\hat{K}_\gamma^h(z_\gamma)) \frac{1}{2\ell \cdot p_i} \mathcal{T}(\hat{K}_\gamma^{-h}(z_\gamma), \hat{p}_j(z_\gamma), \{\beta\}, -\ell) , \quad (2.14)$$

where

$$z_\gamma = -\frac{2p_i \cdot \ell}{2q \cdot \ell} , \quad \hat{K}_\gamma(z_\gamma) = \ell + p_i + z_\gamma q ,$$

and $\{\beta\} = \{1, 2, \dots, n\}/\{i, j\}$.

$\mathcal{R}'''_{B,1}$ denotes the special case when set $\{\beta\} = \emptyset$. In this case, $\mathcal{T}(\hat{K}_\gamma, \hat{p}_j, \{\beta\}, -\ell)$ becomes a three-point amplitude, and we get explicitly

$$\mathcal{R}'''_{B,1} = \sum_h \mathcal{T}(\ell, \hat{p}_i(z_\gamma), \{\gamma\}, -\hat{K}_\gamma^h(z_\gamma)) \frac{1}{-2\ell \cdot p_j} A(\hat{K}_\gamma^{-h}(z_\gamma), \hat{p}_j(z_\gamma), -\ell) , \quad (2.15)$$

where

$$z_\gamma = \frac{2p_j \cdot \ell}{2q \cdot \ell} , \quad \hat{K}_\gamma(z_\gamma) = -(p_j - \ell - z_\gamma q) , \quad (2.16)$$

and $\{\gamma\} = \{1, 2, \dots, n\}/\{i, j\}$.

³We need to distinguish massless bubble from massive bubble. The latter is allowed for one-loop diagrams.

Similarly, we can also organize $\mathcal{R}_{B,2}^{\mathcal{Q}}$ into three parts,

$$\mathcal{R}_{B,2}^{\mathcal{Q}} = \mathcal{R}'_{B,2} + \mathcal{R}''_{B,2} + \mathcal{R}'''_{B,2} , \quad (2.17)$$

just as it is defined for $\mathcal{R}_{B,1}^{\mathcal{Q}}$, but changing $\ell \rightarrow -\ell$. Explicitly, we have

$$\mathcal{R}'_{B,2} = \mathcal{R}'_{B,1}|_{\ell \rightarrow -\ell} , \quad (2.18)$$

and $\mathcal{R}''_{B,2} = \mathcal{R}''_{B,1}|_{\ell \rightarrow -\ell}$, $\mathcal{R}'''_{B,2} = \mathcal{R}'''_{B,1}|_{\ell \rightarrow -\ell}$.

There is an important observation. If we consider the color-ordered integrand, we can choose the deformation pair (i, j) such that $\ell, -\ell$ are not nearly with the deformed momenta. Thus the contributions of $\mathcal{R}''_{B,2}$, $\mathcal{R}''_{B,1}$, $\mathcal{R}'''_{B,2}$ and $\mathcal{R}'''_{B,1}$ do not exist. As we will discuss in the following subsection, the remaining forward singularities that corresponding to massless bubbles are exactly in those four terms. In other words, with a proper choice of deformation pair, we can naturally avoid forward singularities without further using the scale deformation.

2.3 Step three: scale deformation

In the previous subsection we have expressed $\mathcal{T}^{\mathcal{Q}}$ as

$$\mathcal{T}^{\mathcal{Q}} = \mathcal{R}_A^{\mathcal{Q}} + \mathcal{R}_B^{\mathcal{Q}} , \quad (2.19)$$

where $\mathcal{R}_A^{\mathcal{Q}} = \mathcal{R}_{A,1}^{\mathcal{Q}} + \mathcal{R}_{A,2}^{\mathcal{Q}}$ given in expressions (2.7), (2.8) respectively, and $\mathcal{R}_B^{\mathcal{Q}} = \mathcal{R}_{B,1}^{\mathcal{Q}} + \mathcal{R}_{B,2}^{\mathcal{Q}}$, with $\mathcal{R}_{B,1}^{\mathcal{Q}} = \mathcal{R}'_{B,1} + \mathcal{R}''_{B,1} + \mathcal{R}'''_{B,1}$ given in expressions (2.13), (2.14), (2.15), and $\mathcal{R}_{B,2}^{\mathcal{Q}} = \mathcal{R}'_{B,2} + \mathcal{R}''_{B,2} + \mathcal{R}'''_{B,2}$ by changing $\ell \rightarrow -\ell$ of $\mathcal{R}_{B,1}^{\mathcal{Q}}$. In each \mathcal{R} expression there would be \mathcal{T} functions, and we should identify them. The \mathcal{T} functions are determined by removing tree diagrams that corresponding to tadpole and massless bubbles. In the previous subsections, we have presented some discussions on this point, but the complete resolution will be provided in this subsection. In fact, as we have pointed out, the only left forward singularities are those in terms $\mathcal{R}''_{B,1}$, $\mathcal{R}'''_{B,1}$ and $\mathcal{R}''_{B,2}$, $\mathcal{R}'''_{B,2}$. To deal with them, we use the scale deformation.

Before giving a careful discussion, let us take a look on $\mathcal{R}_{A,1}^{\mathcal{Q}}$, $\mathcal{R}_{A,2}^{\mathcal{Q}}$. When multiplying $\frac{1}{\ell^2}$ with \mathcal{T} in (2.7), (2.8), it trivially becomes one-loop integrand of the original \mathcal{Q} -cut representation with BCFW-deformed momenta. Thus we can identify them as

$$\frac{\mathcal{R}_{A,1}^{\mathcal{Q}}}{\ell^2} = \sum_{h,\gamma} A(\widehat{p}_i(z_\gamma), \{\gamma\}, -\widehat{K}_\gamma^h(z_\gamma)) \frac{1}{K_\gamma^2} \mathcal{I}^{\mathcal{Q}}(\widehat{K}_\gamma^{-h}(z_\gamma), \widehat{p}_j(z_\gamma), \{\beta\}, \ell, -\ell) , \quad (2.20)$$

where $z_\gamma = -\frac{(P_\gamma + p_i)^2}{2q \cdot P_\gamma}$, $\widehat{K}_\gamma(z_\gamma) = P_\gamma + p_i + z_\gamma q$. Similarly,

$$\frac{\mathcal{R}_{A,2}^{\mathcal{Q}}}{\ell^2} = \sum_{h,\beta} \mathcal{I}^{\mathcal{Q}}(\ell, -\ell, \{\gamma\}, \widehat{p}_i(z_\beta), -\widehat{K}_\beta^h(z_\beta)) \frac{1}{K_\beta^2} A(\widehat{K}_\beta^{-h}(z_\beta), \{\beta\}, \widehat{p}_j(z_\beta)) , \quad (2.21)$$

where $z_\beta = \frac{(P_\beta + p_j)^2}{2q \cdot P_\beta}$, $\widehat{K}_\beta(z_\beta) = -(P_\beta + p_j + z_\beta q)$. Here $\mathcal{I}^\mathcal{Q}$'s are lower-point one-loop integrands from \mathcal{Q} -cut representation, and A 's are lower-point tree amplitudes. In fact, the one-loop integrand in (2.20) and (2.21) does not need to be in \mathcal{Q} -cut representation, i.e., any representation, such as the one obtained by Feynman diagrams, should be fine. Thus these two terms can be expressed as summation over products of lower-point one-loop integrand and tree amplitude. For other two terms $\mathcal{R}'_{B,1}, \mathcal{R}'_{B,2}$, it has already been shown in (2.13) that they are summation over products of two lower-point tree amplitudes. The important point is that for these two terms, the loop momentum ℓ is not scaled.

Now let us focus on the special cases $\mathcal{R}''_{B,1}, \mathcal{R}'''_{B,1}, \mathcal{R}''_{B,2}, \mathcal{R}'''_{B,2}$, and specifically take $\mathcal{R}''_{B,1}$

$$\mathcal{R}''_{B,1} = \sum_h A(\ell, \widehat{p}_i(z_\gamma), -\widehat{K}_\gamma^h(z_\gamma)) \frac{1}{2\ell \cdot p_i} \mathcal{T}(\widehat{K}_\gamma^{-h}(z_\gamma), \widehat{p}_j(z_\gamma), \{1, \dots, n\}/\{i, j\}, -\ell) \quad (2.22)$$

as example. We need to exclude the contribution of massless bubbles from it. In order to do so, let us introduce a scale deformation $\ell \rightarrow \alpha\ell$ as is done in the original \mathcal{Q} -cut construction. Since $z_\gamma = -\frac{2p_i \cdot \ell}{2q \cdot \ell}$, the scale deformation will not change the location of pole z_γ . Hence we can write $\mathcal{R}''_{B,1}$ as

$$\mathcal{R}''_{B,1}(\alpha) = \sum_h A(\alpha\ell, \widehat{p}_i(z_\gamma), -\widehat{K}_\gamma^h(z_\gamma, \alpha)) \frac{1}{2\ell \cdot p_i} \mathcal{T}(\widehat{K}_\gamma^{-h}(z_\gamma, \alpha), \widehat{p}_j(z_\gamma), \{1, \dots, n\}/\{i, j\}, -\alpha\ell), \quad (2.23)$$

where $\widehat{K}_\gamma(z_\gamma, \alpha) = \alpha\ell + p_i + z_\gamma q$.

Let us have a more detailed discussion on the $\mathcal{T}(\widehat{K}_\gamma, \widehat{p}_j, \{1, \dots, n\}/\{i, j\}, -\alpha\ell)$ of (2.23). The on-shell condition of \widehat{K}_γ is manifestly satisfied for any value of α , since (remembering that $q \cdot p_i = 0$)

$$\widehat{K}_\gamma^2 = (\alpha\ell + p_i - \frac{2p_i \cdot \ell}{2q \cdot \ell} q)^2 = \alpha(2p_i \cdot \ell) - \alpha(2q \cdot \ell) \frac{2p_i \cdot \ell}{2q \cdot \ell} = 0. \quad (2.24)$$

Having verified the on-shell condition, let us concentrate on the pole structure. We will divide poles into three categories. If the pole does not contain $-\alpha\ell$ and \widehat{K}_γ , then it could either be the sum P of some ordinary external legs, or the one containing $\widehat{p}_j = p_j + \frac{2p_i \cdot \ell}{2q \cdot \ell} q$. For the latter case, we have

$$(P + p_j + \frac{2p_i \cdot \ell}{2q \cdot \ell} q)^2 = (P^2 + 2P \cdot p_j) + (2P \cdot q) \frac{2p_i \cdot \ell}{2q \cdot \ell} = \frac{2((P^2 + 2P \cdot p_j)q + (2P \cdot q)p_i) \cdot \ell}{2q \cdot \ell}. \quad (2.25)$$

So this pole is in the scale free form. Similarly, if \widehat{p}_j appears in the numerator, it will give a contribution of $q \cdot \ell$ in the denominator. Anyway it is also in the scale free form. In other words, these poles does not depend on α under the scale deformation.

If the pole contains $-\alpha\ell$ or $\widehat{K}_\gamma = \alpha\ell + \widehat{p}_i$, we can always use momentum conservation to rewrite \widehat{K} as the leg $-\alpha\ell$, so that the pole is in the form containing $-\alpha\ell$. For these cases, we can have either $(P - \alpha\ell)^2 = P^2 - \alpha(2P \cdot \ell)$ leading to a finite pole $\alpha_P = \frac{P^2}{2P \cdot \ell}$, or

$$(P + p_j - z_\gamma q - \alpha\ell)^2 = P^2 + 2P \cdot p_j + (2P \cdot q)z_\gamma - 2\alpha(P + p_j + p_i) \cdot \ell, \quad (2.26)$$

leading to a finite pole

$$\alpha_P = \frac{P^2 + 2P \cdot p_j + (2P \cdot q)z_\gamma}{2(P + p_i + p_j) \cdot \ell}. \quad (2.27)$$

Note that both solutions depend on the loop momentum ℓ .

If the pole contains both $-\alpha\ell$ and \widehat{K} , then it has no dependence on α . This case contains the contribution corresponding to massless bubbles which should be excluded. To see this, let us recall that for the tree diagram that corresponding to massless bubbles with massless external leg \widehat{p}_i , the legs ℓ, \widehat{p}_i are attached to the same three-point vertex, then they meet leg $-\ell$ in the neighboring vertex. Explicitly for the tree diagrams of $\mathcal{T}(\widehat{K}, \widehat{p}_j, \{1, \dots, n\}/\{i, j\}, -\ell)$, it corresponds to the diagrams where legs \widehat{K} and $-\ell$ are attached to the same vertex⁴. This means that the terms corresponding to the massless bubbles are included in the boundary part.

Having understood poles of above three categories, we can now consider the following contour integration

$$\oint \frac{d\alpha}{\alpha-1} \mathcal{T}(\widehat{K}_\gamma(z_\gamma, \alpha), \widehat{p}_j(z_\gamma), \{1, 2, \dots, n\}/\{i, j\}, -\alpha\ell) \\ = \oint \frac{d\alpha}{\alpha-1} \frac{N(-\alpha\ell, \widehat{p}_j)}{\prod_{\lambda_1} (P_{\lambda_1} + p_j - z_\gamma q)^2 \prod_{\lambda_2} (P_{\lambda_2} - \alpha\ell)^2 \prod_{\lambda_3} (P_{\lambda_3} + p_j - z_\gamma q - \alpha\ell)^2}, \quad (2.28)$$

where in the second line we have explicitly written down the above mentioned subtle factors in the denominator. Now we consider its various pole contributions,

- The pole $\alpha = 1$ gives the full un-deformed tree amplitude.
- There are poles at $\alpha = 0$. Such poles will appear for the propagator $(P_{\lambda_2} - \alpha\ell)^2$ when $P_{\lambda_2}^2 = 0$. The other pole $(P_{\lambda_3} + p_j - z_\gamma q - \alpha\ell)^2$ can not contribute to $\alpha = 0$ pole for generic momentum configuration. From expression (2.28) we know that the residue at $\alpha = 0$ is scale free term and we can ignore them. Note that for this argument to be true, we have assumed the factor $A(\alpha\ell, \widehat{p}_i(z_\gamma), -\widehat{K}_\gamma(z_\gamma, \alpha))$ in (2.23) would not provide denominator that breaking the scale free form.
- For the pole at $\alpha = \infty$, it contains the contribution from massless bubbles, which should be excluded. However, It also contains other contributions which should be included in the final result. But inspecting the expression (2.28), it can be checked that all such contributions are scale free terms, and we can exclude all the contributions at $\alpha = \infty$, letting the result to be valid up to some scale free terms.

With above consideration, we can claim that, the contributions of finite α poles are the ones we need for constructing the one-loop integrands, without the contributions that corresponding to tadpole and massless bubbles, and valid up to some scale free terms. Thus we can write $\mathcal{T}(\widehat{K}_\gamma^h(z_\gamma, \alpha), \widehat{p}_j(z_\gamma), \{1, 2, \dots, n\}/\{i, j\}, -\alpha\ell)$ as

$$\mathcal{T} = \sum_{h', \lambda \in P_\lambda^2 \neq 0} A(\widehat{K}_\gamma^h(z_\gamma, \alpha_\lambda), \{\beta\}, K_\lambda^{-h'}(\alpha_\lambda)) \frac{1}{P_\lambda^2 - 2P_\lambda \cdot \ell} A(-K_\lambda^{h'}(\alpha_\lambda), \{\lambda\}, -\alpha_\lambda\ell), \quad (2.29)$$

⁴It is easy to see that if we perform the scale deformation $\ell \rightarrow \alpha\ell$, such terms will not contain α in the denominator.

where $\alpha_\lambda = \frac{P_\lambda^2}{2P_\lambda \cdot \ell}$, $K_\lambda(\alpha_\lambda) = P_\lambda - \alpha_\lambda \ell$, $\{\beta\} \cup \{\lambda\} = \{1, 2, \dots, n\}/\{i, j\} + \{\widehat{j}\}$, and the summation is over all possible splitting of $\{1, 2, \dots, n\}/\{i, j\} + \{\widehat{j}\}$, but with the condition $P_\lambda^2 = 0$, which means that the set $\{\lambda\}$ should have more than one external leg.

With above result, we can finally write the $\mathcal{R}_{B,1}''$ as

$$\mathcal{R}_{B,1}'' = \sum_h A(\ell, \widehat{p}_i, -\widehat{K}_\gamma^h) \frac{1}{2\ell \cdot p_i} \left(\sum_{h', \lambda \in P_\lambda^2 \neq 0} A(\widehat{K}_\gamma^{-h}, \{\beta\}, K_\lambda^{-h'}) \frac{1}{P_\lambda^2 - 2P_\lambda \cdot \ell} A(-K_\lambda^{h'}, \{\lambda\}, -\alpha_\lambda \ell) \right), \quad (2.30)$$

where

$$z_\gamma = -\frac{2p_i \cdot \ell}{2q \cdot \ell}, \quad \alpha_\lambda = \frac{P_\lambda^2}{2P_\lambda \cdot \ell},$$

and $\widehat{p}_i = p_i + z_\gamma q$, $\widehat{K}_\gamma = \alpha_\lambda \ell + p_i + z_\gamma q$, $K_\lambda = P_\lambda - \alpha_\lambda \ell$, $\{\beta\} \cup \{\lambda\} = \{1, 2, \dots, n\}/\{i, j\} + \{\widehat{j}\}$.

Similarly, we have

$$\mathcal{R}_{B,1}''' = \sum_h \left(\sum_{h', \lambda \in P_\lambda^2 \neq 0} A(\alpha_\lambda \ell, \{\lambda\}, -K_\lambda^{h'}) \frac{1}{P_\lambda^2 + 2P_\lambda \cdot \ell} A(K_\lambda^{-h'}, \{\beta\}, -\widehat{K}_\gamma^h) \right) \frac{1}{-2\ell \cdot p_j} A(\widehat{K}_\gamma^{-h}, \widehat{p}_j, -\ell) \quad (2.31)$$

where

$$z_\gamma = \frac{2p_j \cdot \ell}{2q \cdot \ell}, \quad \alpha_\lambda = -\frac{P_\lambda^2}{2P_\lambda \cdot \ell},$$

and $\widehat{p}_j = p_j - z_\gamma q$, $K_\lambda = P_\lambda + \alpha_\lambda \ell$, $K_\gamma = -\alpha_\lambda \ell + p_j - z_\gamma q$, $\{\lambda\} \cup \{\beta\} = \{1, 2, \dots, n\}/\{i, j\} + \{\widehat{i}\}$.

We also have

$$\mathcal{R}_{B,2}'' = \mathcal{R}_{B,1}''|_{\ell \rightarrow -\ell}, \quad \mathcal{R}_{B,2}''' = \mathcal{R}_{B,1}'''|_{\ell \rightarrow -\ell}. \quad (2.32)$$

To summarize, by BCFW deformation, we have expressed the n -point one-loop integrand recursively as

$$\mathcal{I}_n = \frac{1}{\ell^2} (\mathcal{R}_A^Q + \mathcal{R}_B^Q), \quad (2.33)$$

where $\mathcal{R}_A^Q = \mathcal{R}_{A,1}^Q + \mathcal{R}_{A,2}^Q$, and $\frac{1}{\ell^2} \mathcal{R}_{A,1}^Q$, $\frac{1}{\ell^2} \mathcal{R}_{A,2}^Q$ are defined as formulas (2.20), (2.21) respectively, which are summation of products of lower-point tree amplitude with low-point one-loop integrand of Q -cut construction. Also, $\mathcal{R}_B^Q = \mathcal{R}_{B,1}' + \mathcal{R}_{B,1}'' + \mathcal{R}_{B,1}''' + \mathcal{R}_{B,2}' + \mathcal{R}_{B,2}'' + \mathcal{R}_{B,2}'''$. Among which, $\mathcal{R}_{B,1}'$, $\mathcal{R}_{B,2}'$ are defined in formulas (2.13), (2.18) respectively, which are summation of products of two lower-point tree amplitudes, and $\mathcal{R}_{B,1}''$, $\mathcal{R}_{B,1}'''$, $\mathcal{R}_{B,2}''$, $\mathcal{R}_{B,2}'''$ are defined in formulas (2.30), (2.31), (2.32) respectively, which are although products of three lower-point tree amplitudes, but one of them is the three-point amplitude. It is also important to notice how the forward singularities have been removed in various terms by various methods.

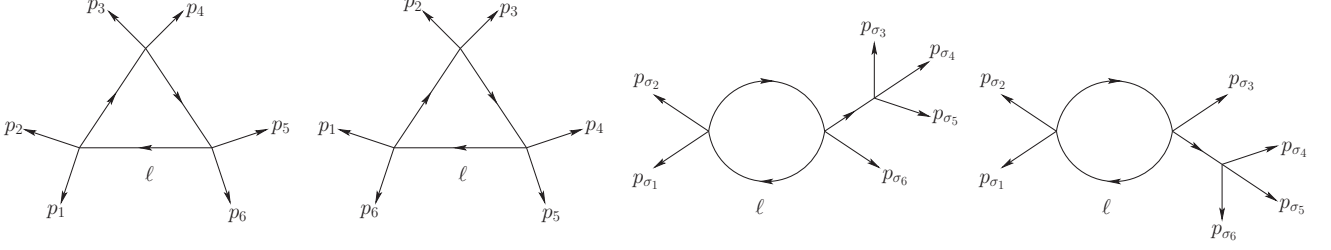


Figure 1. Feynman diagrams of color-ordered one-loop six-point amplitude in scalar ϕ^4 theory. There are two triangle diagrams and twelve bubble diagrams with $\{\sigma_1, \dots, \sigma_6\} \in \text{Cyclic}\{1, 2, 3, 4, 5, 6\}$.

3 Some examples

In the previous section, we have presented a recursive formula for one-loop integrand construction, based on the BCFW deformation and \mathcal{Q} -cut construction. This new construction shows that there are other ways to write down a well-defined one-loop integrand which is valid up to scale free terms. The recursive formula (2.33) has given an alternative factorization of one-loop integrand, and it should be equivalent to the result of original \mathcal{Q} -cut representation or Feynman diagram method, at least up to some scale free terms. For a better understanding of this recursive formula, in this section, we shall present detailed computation of some one-loop integrands by recursive formula (2.33), and demonstrate their correspondence with results of original \mathcal{Q} -cut construction and Feynman diagram methods.

3.1 The one-loop six-point amplitude in scalar ϕ^4 theory

In this example we consider the integrand of one-loop six-point amplitude in color ordered scalar ϕ^4 theory. For this theory, there is no cubic vertex, so the computation is relatively simple since we do not need to use the scale deformation to remove singular terms. After using appropriate BCFW deformation to get rid of boundary contribution, we need to consider contributions from all detectable finite poles of both $\mathcal{R}_A^{\mathcal{Q}}$ and $\mathcal{R}_B^{\mathcal{Q}}$. In order to verify the equivalence term by term, we will compute the integrand by Feynman diagram method, the original \mathcal{Q} -cut representation and the recursive formula (2.33).

Feynman diagram method: there are in total fourteen Feynman diagrams as shown in Figure 1. Using the Feynman rules, we directly get

$$\begin{aligned} \mathcal{I}^{\mathcal{F}} = & \frac{1}{\ell^2(\ell - p_{12})^2(\ell - p_{1234})^2} + \frac{1}{\ell^2(\ell - p_{61})^2(\ell - p_{6123})^2} \\ & + \frac{1}{\ell^2(\ell - p_{\sigma_1\sigma_2})^2} \frac{1}{p_{\sigma_3\sigma_4\sigma_5}^2} + \frac{1}{\ell^2(\ell - p_{\sigma_1\sigma_2})^2} \frac{1}{p_{\sigma_4\sigma_5\sigma_6}^2} \quad \text{for } \sigma \in \text{Cyclic}\{1, 2, 3, 4, 5, 6\}. \end{aligned} \quad (3.1)$$

Applying the partial fraction identity

$$\frac{1}{D_1 \cdots D_m} = \sum_{i=1}^m \frac{1}{D_i} \left[\prod_{j \neq i} \frac{1}{D_j - D_i} \right], \quad (3.2)$$

we can rewrite above result as

$$\begin{aligned} \mathcal{I}^{\mathcal{F}} = & \left\{ \frac{1}{\ell^2(-2\ell \cdot p_{12} + p_{12}^2)(-2\ell \cdot p_{1234} + p_{1234}^2)} + \text{Cyclic}\{1, 2, 3, 4, 5, 6\} \right\} \\ & + \left\{ \left(\frac{1}{\ell^2(-2\ell \cdot p_{12} + p_{12}^2)} + \frac{1}{\ell^2(-2\ell \cdot p_{3456} + p_{3456}^2)} \right) \left(\frac{1}{p_{345}^2} + \frac{1}{p_{456}^2} \right) + \text{Cyclic}\{1, 2, 3, 4, 5, 6\} \right\}. \end{aligned} \quad (3.3)$$

When expanded, the first line contains 6 terms from triangle diagrams, and the second line contains $4 \times 6 = 24$ terms from bubble diagrams.

The \mathcal{Q} -cut representation: the integrand is given by

$$\begin{aligned} \mathcal{I}^{\mathcal{Q}} = & \mathcal{A}_4(1, 2, \widehat{\ell}_R, -\widehat{\ell}_L) \frac{1}{\ell^2(-2\ell \cdot p_{12} + p_{12}^2)} \mathcal{A}_6(\widehat{\ell}_L, -\widehat{\ell}_R, 3, 4, 5, 6) \Big|_{\widehat{\ell}=\alpha_{12}\ell} + \text{Cyclic}\{1, 2, 3, 4, 5, 6\} \\ & + \mathcal{A}_6(1, 2, 3, 4, \widehat{\ell}_R, -\widehat{\ell}_L) \frac{1}{\ell^2(-2\ell \cdot p_{1234} + p_{1234}^2)} \mathcal{A}_4(\widehat{\ell}_L, -\widehat{\ell}_R, 5, 6) \Big|_{\widehat{\ell}=\alpha_{1234}\ell} + \text{Cyclic}\{1, 2, 3, 4, 5, 6\}, \end{aligned}$$

where $\alpha_{i_1 i_2} = \frac{p_{i_1 i_2}^2}{2\ell \cdot p_{i_1 i_2}}$, $\alpha_{i_1 i_2 i_3 i_4} = \frac{p_{i_1 i_2 i_3 i_4}^2}{2\ell \cdot p_{i_1 i_2 i_3 i_4}}$ and $\ell^2 = 0$. The six-point tree-level amplitude in general dimension is

$$\mathcal{A}_6(1, 2, 3, 4, 5, 6) = \frac{1}{p_{123}^2} + \frac{1}{p_{234}^2} + \frac{1}{p_{345}^2}. \quad (3.4)$$

Inserting it back to above expression and rearranging some terms by cyclic invariance, we get explicitly

$$\begin{aligned} \mathcal{I}^{\mathcal{Q}} = & \left\{ \left(\frac{2\ell \cdot p_{12}}{\ell^2(-2\ell \cdot p_{12} + p_{12}^2)} - \frac{2\ell \cdot p_{1234}}{\ell^2(-2\ell \cdot p_{1234} + p_{1234}^2)} \right) \frac{1}{-(2\ell \cdot p_{1234})p_{12}^2 + (2\ell \cdot p_{12})p_{1234}^2} + \text{Cyclic}\{1, 2, 3, 4, 5, 6\} \right\} \\ & + \left\{ \left(\frac{1}{\ell^2(-2\ell \cdot p_{12} + p_{12}^2)} + \frac{1}{\ell^2(-2\ell \cdot p_{1234} + p_{1234}^2)} \right) \left(\frac{1}{p_{345}^2} + \frac{1}{p_{456}^2} \right) + \text{Cyclic}\{1, 2, 3, 4, 5, 6\} \right\}. \end{aligned} \quad (3.5)$$

The second line contains 24 terms, which is identical to the second line of result (3.3) by Feynman diagram method. The first line contains 12 terms and can be organized as 6 pairs. The sum of each pair leads to

$$\begin{aligned} & \frac{(2\ell \cdot p_{12})(-2\ell \cdot p_{1234} + p_{1234}^2) - (2\ell \cdot p_{1234})(-2\ell \cdot p_{12} + p_{12}^2)}{\ell^2(-2\ell \cdot p_{12} + p_{12}^2)(-2\ell \cdot p_{1234} + p_{1234}^2)} \frac{1}{-(2\ell \cdot p_{1234})p_{12}^2 + (2\ell \cdot p_{12})p_{1234}^2} + \cdots \\ = & \frac{1}{\ell^2(-2\ell \cdot p_{12} + p_{12}^2)(-2\ell \cdot p_{1234} + p_{1234}^2)} + \text{Cyclic}\{1, 2, 3, 4, 5, 6\}, \end{aligned} \quad (3.6)$$

which equals to the 6 terms in the first line of result (3.3) by Feynman diagram method.

Recursive formula: now let us discuss the recursive construction of $\mathcal{T}^{\mathcal{Q}}$ and the integrand $\mathcal{I} = \frac{1}{\ell^2} \mathcal{T}^{\mathcal{Q}}$. Because of the ϕ^4 theory, in this example only $\mathcal{R}_{A,1}^{\mathcal{Q}}, \mathcal{R}_{A,2}^{\mathcal{Q}}$ and $\mathcal{R}'_{B,1}, \mathcal{R}'_{B,2}$ will contribute to the final integrand, while the contributions $\mathcal{R}''_{B,1}, \mathcal{R}'''_{B,1}, \mathcal{R}''_{B,2}, \mathcal{R}'''_{B,2}$ are vanishing since the three-point amplitude vanishes. Since we are considering color-ordered amplitude, $\mathcal{T}^{\mathcal{Q}}$ will be the sum of six diagrams,

$$\begin{aligned} \mathcal{T}^{\mathcal{Q}} = & \mathcal{T}_1^{\mathcal{Q}}(\ell, -\ell, 1, 2, 3, 4, 5, 6) + \mathcal{T}_2^{\mathcal{Q}}(\ell, -\ell, 2, 3, 4, 5, 6, 1) + \mathcal{T}_3^{\mathcal{Q}}(\ell, -\ell, 3, 4, 5, 6, 1, 2) \\ & + \mathcal{T}_4^{\mathcal{Q}}(\ell, -\ell, 4, 5, 6, 1, 2, 3) + \mathcal{T}_5^{\mathcal{Q}}(\ell, -\ell, 5, 6, 1, 2, 3, 4) + \mathcal{T}_6^{\mathcal{Q}}(\ell, -\ell, 6, 1, 2, 3, 4, 5) , \end{aligned} \quad (3.7)$$

where in each diagram, one internal line has been cut. In order to avoid boundary contribution, the two momenta to be deformed should at least be separated by two legs. So we can take the BCFW deformation as

$$\widehat{p}_1 = p_1 + zq \quad , \quad \widehat{p}_4 = p_4 - zq \quad , \quad q^2 = p_{1,4} \cdot q = 0 \quad . \quad (3.8)$$

Note that we are not necessary to take the same deformation for all $\mathcal{T}_i^{\mathcal{Q}}$'s. In the practical computation, we can take the most convenient BCFW deformation for each $\mathcal{T}_i^{\mathcal{Q}}$. But here we use the same deformation for demonstration. Under this deformation, we then compute the non-vanishing BCFW terms for each $\mathcal{T}_i^{\mathcal{Q}}$. Let us define

$$z_{123} \equiv -\frac{p_{123}^2}{2q \cdot p_{123}} \quad , \quad z_{561} \equiv -\frac{p_{561}^2}{2q \cdot p_{561}} \quad , \quad z_{612} \equiv -\frac{p_{612}^2}{2q \cdot p_{612}} \quad , \quad z_{12}^{\pm} \equiv -\frac{\pm 2\ell \cdot p_{12} + p_{12}^2}{2q \cdot (p_{12} \pm \ell)} \quad , \quad (3.9)$$

$$z_{34}^{\pm} \equiv -\frac{\pm 2\ell \cdot p_{34} + p_{34}^2}{2q \cdot (p_{34} \pm \ell)} \quad , \quad z_{45}^{\pm} \equiv -\frac{\pm 2\ell \cdot p_{45} + p_{45}^2}{2q \cdot (p_{45} \pm \ell)} \quad , \quad z_{61}^{\pm} \equiv -\frac{\pm 2\ell \cdot p_{61} + p_{61}^2}{2q \cdot (p_{61} \pm \ell)} \quad . \quad (3.10)$$

For tree diagram of $\mathcal{T}_1^{\mathcal{Q}}$, there would be five contributing terms under this deformation. The first is a $\mathcal{R}_{A,2}^{\mathcal{Q}}$ -type contribution,

$$\begin{aligned} \mathcal{T}_{11}^{\mathcal{Q}} = & \mathcal{A}_4(\widehat{1}, 2, \widehat{\ell}_R, -\widehat{\ell}_L) \frac{1}{-2\ell \cdot p_{\widehat{1}2} + p_{12}^2} \mathcal{A}_4(\widehat{\ell}_L, -\widehat{\ell}_R, 3, \widehat{P}) \frac{1}{p_{123}^2} \mathcal{A}_4(-\widehat{P}, \widehat{4}, 5, 6) \\ = & \frac{1}{-2\ell \cdot p_{\widehat{1}2} + p_{12}^2} \frac{1}{p_{123}^2} \Big|_{z_{123}} \quad , \end{aligned} \quad (3.11)$$

where \widehat{P} is understood to follow the momentum conservation of each sub-amplitude, and $z = z_{123}$, $\alpha = \frac{p_{12}^2}{2\ell \cdot p_{\widehat{1}2}} \Big|_{z_{123}}$. The second is a $\mathcal{R}_{A,1}^{\mathcal{Q}}$ -type contribution,

$$\begin{aligned} \mathcal{T}_{12}^{\mathcal{Q}} = & \mathcal{A}_4(\widehat{1}, 2, 3, \widehat{P}) \frac{1}{p_{123}^2} \mathcal{A}_4(-\widehat{P}, \widehat{4}, \widehat{\ell}_R, -\widehat{\ell}_L) \frac{1}{-2\ell \cdot p_{\widehat{P}4} + p_{\widehat{P}4}^2} \mathcal{A}_4(\widehat{\ell}_L, -\widehat{\ell}_R, 5, 6) \\ = & \frac{1}{p_{123}^2} \frac{1}{2\ell \cdot p_{56} + p_{56}^2} = \frac{1}{-2\ell \cdot p_{1234} + p_{1234}^2} \frac{1}{p_{123}^2} \quad , \end{aligned} \quad (3.12)$$

where $z = z_{123}$, $\alpha = -\frac{p_{56}^2}{2\ell \cdot p_{56}}$. The third is a $\mathcal{R}_{A,2}^{\mathcal{Q}}$ -type contribution,

$$\begin{aligned}\mathcal{T}_{13}^{\mathcal{Q}} &= \mathcal{A}_4(\widehat{1}, 2, \widehat{\ell}_R, -\widehat{\ell}_L) \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \mathcal{A}_4(\widehat{\ell}_L, -\widehat{\ell}_R, \widehat{P}, 6) \frac{1}{p_{612}^2} \mathcal{A}_4(-\widehat{P}, 3, \widehat{4}, 5) \\ &= \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \frac{1}{p_{612}^2} \Big|_{z_{612}},\end{aligned}\quad (3.13)$$

where $z = z_{612}$, $\alpha = \frac{p_{12}^2}{2\ell \cdot p_{12}} \Big|_{z_{612}}$. The fourth is a $\mathcal{R}_{A,1}^{\mathcal{Q}}$ -type contribution,

$$\begin{aligned}\mathcal{T}_{14}^{\mathcal{Q}} &= \mathcal{A}_4(\widehat{1}, \widehat{P}, \widehat{\ell}_R, -\widehat{\ell}_L) \frac{1}{-2\ell \cdot p_{1\widehat{P}} + p_{1\widehat{P}}^2} \mathcal{A}_4(\widehat{\ell}_L, -\widehat{\ell}_R, 5, 6) \frac{1}{p_{561}^2} \mathcal{A}_4(-\widehat{P}, 2, 3, \widehat{4}) \\ &= \frac{1}{2\ell \cdot p_{56} + p_{56}^2} \frac{1}{p_{561}^2} = \frac{1}{-2\ell \cdot p_{1234} + p_{1234}^2} \frac{1}{p_{561}^2},\end{aligned}\quad (3.14)$$

where $z = z_{561}$, $\alpha = -\frac{p_{56}^2}{2\ell \cdot p_{56}}$. Finally, the fifth is a $\mathcal{R}'_{B,2}$ contribution,

$$\begin{aligned}\mathcal{T}_{15}^{\mathcal{Q}} &= \mathcal{A}_4(-\ell, \widehat{1}, 2, \widehat{P}) \frac{1}{(p_{12} - \ell)^2} \mathcal{A}_6(-\widehat{P}, 3, \widehat{4}, 5, 6, \ell) = \frac{1}{(p_{12} - \ell)^2} \left(\frac{1}{p_{345}^2} + \frac{1}{p_{456}^2} + \frac{1}{(\ell + p_{56})^2} \right) \Big|_{\ell^2=0, z=z_{12}^-} \\ &= \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \frac{1}{p_{345}^2} \Big|_{z_{12}^-} + \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \frac{1}{p_{456}^2} \Big|_{z_{12}^-} + \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \frac{1}{-2\ell \cdot p_{1234} + p_{1234}^2} \\ &\equiv \mathcal{T}_{15,1}^{\mathcal{Q}} + \mathcal{T}_{15,2}^{\mathcal{Q}} + \mathcal{T}_{15,3}^{\mathcal{Q}},\end{aligned}\quad (3.15)$$

where $z = z_{12}^-$.

So for $\mathcal{T}_1^{\mathcal{Q}}$, in total we get seven terms. Let us see how these seven terms is corresponding to the terms in \mathcal{Q} -cut representation. $\mathcal{T}_{12}^{\mathcal{Q}}$, $\mathcal{T}_{14}^{\mathcal{Q}}$ and $\mathcal{T}_{15,3}^{\mathcal{Q}}$ are evaluated with the un-deformed momenta. It is simple to see that $\frac{1}{\ell^2} \mathcal{T}_{15,3}^{\mathcal{Q}}$ corresponds to a term in the first line of (3.3), while $\frac{1}{\ell^2} \mathcal{T}_{12}^{\mathcal{Q}}$, $\frac{1}{\ell^2} \mathcal{T}_{14}^{\mathcal{Q}}$ also have their equivalent terms in the second line of (3.3),

$$\frac{1}{\ell^2} (\mathcal{T}_{12}^{\mathcal{Q}} + \mathcal{T}_{14}^{\mathcal{Q}}) = \frac{1}{\ell^2 (-2\ell \cdot p_{1234} + p_{1234}^2)} \left(\frac{1}{p_{123}^2} + \frac{1}{p_{234}^2} \right). \quad (3.16)$$

There are also four terms $\mathcal{T}_{11}^{\mathcal{Q}}$, $\mathcal{T}_{13}^{\mathcal{Q}}$, $\mathcal{T}_{15,1}^{\mathcal{Q}}$, $\mathcal{T}_{15,2}^{\mathcal{Q}}$ evaluated with deformed momenta. We have

$$\begin{aligned}\mathcal{T}_{11}^{\mathcal{Q}} + \mathcal{T}_{15,2}^{\mathcal{Q}} &= \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \frac{1}{p_{456}^2} \Big|_{z_{123}} + \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \frac{1}{p_{456}^2} \Big|_{z_{12}^-} \\ &= \frac{1}{p_{456}^2} \frac{1}{(-2\ell \cdot p_{12} + p_{12}^2) + \frac{(2q \cdot p_{12} - 2q \cdot \ell)}{2q \cdot p_{456}} p_{456}^2} + \frac{1}{(-2\ell \cdot p_{12} + p_{12}^2)} \frac{1}{p_{456}^2 + \frac{2q \cdot p_{456}}{(2q \cdot p_{12} - 2q \cdot \ell)} (-2\ell \cdot p_{12} + p_{12}^2)},\end{aligned}$$

as well as

$$\begin{aligned}\mathcal{T}_{13}^{\mathcal{Q}} + \mathcal{T}_{15,1}^{\mathcal{Q}} &= \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \frac{1}{p_{345}^2} \Big|_{z_{612}} + \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \frac{1}{p_{345}^2} \Big|_{z_{12}^-} \\ &= \frac{1}{p_{345}^2} \frac{1}{(-2\ell \cdot p_{12} + p_{12}^2) + \frac{(2q \cdot p_{12} - 2q \cdot \ell)}{2q \cdot p_{345}} p_{345}^2} + \frac{1}{(-2\ell \cdot p_{12} + p_{12}^2)} \frac{1}{p_{345}^2 + \frac{2q \cdot p_{345}}{(2q \cdot p_{12} - 2q \cdot \ell)} (-2\ell \cdot p_{12} + p_{12}^2)}.\end{aligned}$$

Using the identity

$$\frac{1}{A(B - \lambda A)} + \frac{1}{B(A - \frac{1}{\lambda}B)} = \frac{1}{AB}, \quad (3.17)$$

we arrive at

$$\frac{1}{\ell^2}(\mathcal{T}_{11}^{\mathcal{Q}} + \mathcal{T}_{15,2}^{\mathcal{Q}} + \mathcal{T}_{13}^{\mathcal{Q}} + \mathcal{T}_{15,1}^{\mathcal{Q}}) = \frac{1}{\ell^2(-2\ell \cdot p_{12} + p_{12}^2)} \left(\frac{1}{p_{345}^2} + \frac{1}{p_{456}^2} \right). \quad (3.18)$$

The above computation shows the one-to-one correspondence between the results of Feynman diagram method and the recursive formula. The contribution of $\frac{1}{\ell^2} \mathcal{T}_1^{\mathcal{Q}}$ is equivalent to the terms in (3.3) with a specific cyclic permutation.

Similarly, we can also check the equivalence of the other five $\mathcal{T}_i^{\mathcal{Q}}$ with the terms in (3.3) of the other cyclic permutation. For tree diagram of $\mathcal{T}_2^{\mathcal{Q}}$, there would also be five contributing terms. The first is a $\mathcal{R}_{A,2}^{\mathcal{Q}}$ -type contribution,

$$\mathcal{T}_{21}^{\mathcal{Q}} = \mathcal{A}_4(2, 3, \hat{\ell}_R, -\hat{\ell}_L) \frac{1}{-2\ell \cdot p_{23} + p_{23}^2} \mathcal{A}_4(\hat{\ell}_L, -\hat{\ell}_R, \hat{P}, \hat{1}) \frac{1}{p_{123}^2} \mathcal{A}_4(-\hat{P}, \hat{4}, 5, 6) = \frac{1}{-2\ell \cdot p_{23} + p_{23}^2} \frac{1}{p_{123}^2},$$

where $z = z_{123}$, $\alpha = \frac{p_{23}^2}{2\ell \cdot p_{23}}$. The second is a $\mathcal{R}_{A,2}^{\mathcal{Q}}$ -type contribution,

$$\mathcal{T}_{22}^{\mathcal{Q}} = \mathcal{A}_4(2, \hat{P}, \hat{\ell}_R, -\hat{\ell}_L) \frac{1}{-2\ell \cdot p_{2\hat{P}} + p_{2\hat{P}}^2} \mathcal{A}_4(\hat{\ell}_L, -\hat{\ell}_R, 6, \hat{1}) \frac{1}{p_{612}^2} \mathcal{A}_4(-\hat{P}, 3, \hat{4}, 5) = \frac{1}{-2\ell \cdot p_{23\hat{4}5} + p_{23\hat{4}5}^2} \frac{1}{p_{612}^2} \Big|_{z_{612}},$$

where $z = z_{612}$, $\alpha = -\frac{p_{61}^2}{2\ell \cdot p_{61}} \Big|_{z_{612}}$. The third is a $\mathcal{R}_{A,2}^{\mathcal{Q}}$ -type contribution,

$$\mathcal{T}_{23}^{\mathcal{Q}} = \mathcal{A}_4(\hat{P}, 5, \hat{\ell}_R, -\hat{\ell}_L) \frac{1}{-2\ell \cdot p_{5\hat{P}} + p_{5\hat{P}}^2} \mathcal{A}_4(\hat{\ell}_L, -\hat{\ell}_R, 6, \hat{1}) \frac{1}{p_{561}^2} \mathcal{A}_4(-\hat{P}, 2, 3, \hat{4}) = \frac{1}{-2\ell \cdot p_{23\hat{4}5} + p_{23\hat{4}5}^2} \frac{1}{p_{561}^2} \Big|_{z_{561}},$$

where $z = z_{561}$, $\alpha = -\frac{p_{61}^2}{2\ell \cdot p_{61}} \Big|_{z_{561}}$. The fourth is a $\mathcal{R}_{A,1}^{\mathcal{Q}}$ -type contribution,

$$\mathcal{T}_{24}^{\mathcal{Q}} = \mathcal{A}_4(5, 6, \hat{1}, \hat{P}) \frac{1}{p_{561}^2} \mathcal{A}_4(2, 3, \hat{\ell}_R, -\hat{\ell}_L) \frac{1}{-2\ell \cdot p_{23} + p_{23}^2} \mathcal{A}_4(\hat{\ell}_L, -\hat{\ell}_R, \hat{4}, -\hat{P}) = \frac{1}{p_{561}^2} \frac{1}{-2\ell \cdot p_{23} + p_{23}^2},$$

where $z = z_{561}$, $\alpha = \frac{p_{23}^2}{2\ell \cdot p_{23}}$. Finally the fifth is a $\mathcal{R}'_{B,1}$ -type contribution,

$$\begin{aligned} \mathcal{T}_{25}^{\mathcal{Q}} &= \mathcal{A}_4(6, \hat{1}, \ell, \hat{P}) \frac{1}{(p_{61} + \ell)^2} \mathcal{A}_6(-\hat{P}, -\ell, 2, 3, \hat{4}, 5) = \frac{1}{(p_{61} + \ell)^2} \left(\frac{1}{p_{23\hat{4}}^2} + \frac{1}{p_{3\hat{4}5}^2} + \frac{1}{(-\ell + p_{23})^2} \right) \Big|_{\ell^2=0, z=z_{61}^+} \\ &= \frac{1}{-2\ell \cdot p_{23\hat{4}5} + p_{23\hat{4}5}^2} \frac{1}{p_{23\hat{4}}^2} \Big|_{z_{61}^+} + \frac{1}{-2\ell \cdot p_{23\hat{4}5} + p_{23\hat{4}5}^2} \frac{1}{p_{3\hat{4}5}^2} \Big|_{z_{61}^+} + \frac{1}{-2\ell \cdot p_{23\hat{4}5} + p_{23\hat{4}5}^2} \frac{1}{-2\ell \cdot p_{23} + p_{23}^2} \\ &\equiv \mathcal{T}_{25,1}^{\mathcal{Q}} + \mathcal{T}_{25,2}^{\mathcal{Q}} + \mathcal{T}_{25,3}^{\mathcal{Q}}, \end{aligned}$$

where $z = z_{61}^+$.

For tree diagrams of $\mathcal{T}_3^{\mathcal{Q}}$, there are in total six contributing terms. The first is a $\mathcal{R}_{A,2}^{\mathcal{Q}}$ -type contribution,

$$\mathcal{T}_{31}^{\mathcal{Q}} = \mathcal{A}_4(3, \hat{P}, \hat{\ell}_R, -\hat{\ell}_L) \frac{1}{-2\ell \cdot p_{3\hat{P}} + p_{3\hat{P}}^2} \mathcal{A}_4(\hat{\ell}_L, -\hat{\ell}_R, \hat{1}, 2) \frac{1}{p_{123}^2} \mathcal{A}_4(-\hat{P}, \hat{4}, 5, 6) = \frac{1}{-2\ell \cdot p_{3\hat{4}56} + p_{3\hat{4}56}^2} \frac{1}{p_{123}^2} \Big|_{z_{123}},$$

where $z = z_{123}$, $\alpha = -\frac{p_{12}^2}{2\ell \cdot p_{12}} \Big|_{z_{123}}$. The second is a $\mathcal{R}_{A,2}^{\mathcal{Q}}$ -type contribution,

$$\mathcal{T}_{32}^{\mathcal{Q}} = \mathcal{A}_4(\widehat{P}, 6, \widehat{\ell}_R, -\widehat{\ell}_L) \frac{1}{-2\ell \cdot p_{\widehat{P}6} + p_{\widehat{P}6}^2} \mathcal{A}_4(\widehat{\ell}_L, -\widehat{\ell}_R, \widehat{1}, 2) \frac{1}{p_{612}^2} \mathcal{A}_4(-\widehat{P}, 3, \widehat{4}, 5) = \frac{1}{-2\ell \cdot p_{3456} + p_{3456}^2} \frac{1}{p_{612}^2} \Big|_{z_{612}},$$

where $z = z_{612}$, $\alpha = -\frac{p_{12}^2}{2\ell \cdot p_{12}} \Big|_{z_{612}}$. The third is a $\mathcal{R}_{A,1}^{\mathcal{Q}}$ -type contribution,

$$\mathcal{T}_{33}^{\mathcal{Q}} = \mathcal{A}_4(6, \widehat{1}, 2, \widehat{P}) \frac{1}{p_{612}^2} \mathcal{A}_4(3, \widehat{4}, \widehat{\ell}_R, -\widehat{\ell}_L) \frac{1}{-2\ell \cdot p_{34} + p_{34}^2} \mathcal{A}_4(\widehat{\ell}_L, -\widehat{\ell}_R, 5, -\widehat{P}) = \frac{1}{p_{612}^2} \frac{1}{-2\ell \cdot p_{34} + p_{34}^2} \Big|_{z_{612}},$$

where $z = z_{612}$, $\alpha = \frac{p_{34}^2}{2\ell \cdot p_{34}} \Big|_{z_{612}}$. The fourth is a $\mathcal{R}_{A,1}^{\mathcal{Q}}$ -type contribution,

$$\mathcal{T}_{34}^{\mathcal{Q}} = \mathcal{A}_4(5, 6, \widehat{1}, \widehat{P}) \frac{1}{p_{561}^2} \mathcal{A}_4(3, \widehat{4}, \widehat{\ell}_R, -\widehat{\ell}_L) \frac{1}{-2\ell \cdot p_{34} + p_{34}^2} \mathcal{A}_4(\widehat{\ell}_L, -\widehat{\ell}_R, -\widehat{P}, 2) = \frac{1}{p_{561}^2} \frac{1}{-2\ell \cdot p_{34} + p_{34}^2} \Big|_{z_{561}},$$

where $z = z_{561}$, $\alpha = \frac{p_{34}^2}{2\ell \cdot p_{34}} \Big|_{z_{561}}$. The fifth is a $\mathcal{R}'_{B,1}$ -type contribution,

$$\begin{aligned} \mathcal{T}_{35}^{\mathcal{Q}} &= \mathcal{A}_4(\widehat{1}, 2, \ell, \widehat{P}) \frac{1}{(p_{12} + \ell)^2} \mathcal{A}_6(-\widehat{P}, -\ell, 3, \widehat{4}, 5, 6) = \frac{1}{(p_{12} + \ell)^2} \left(\frac{1}{p_{345}^2} + \frac{1}{p_{456}^2} + \frac{1}{(-\ell + p_{34})^2} \right) \Big|_{\ell^2=0, z=z_{12}^+} \\ &= \frac{1}{-2\ell \cdot p_{3456} + p_{3456}^2} \frac{1}{p_{345}^2} \Big|_{z_{12}^+} + \frac{1}{-2\ell \cdot p_{3456} + p_{3456}^2} \frac{1}{p_{456}^2} \Big|_{z_{12}^+} + \frac{1}{-2\ell \cdot p_{3456} + p_{3456}^2} \frac{1}{-2\ell \cdot p_{34} + p_{34}^2} \Big|_{z_{12}^+} \\ &\equiv \mathcal{T}_{35,1}^{\mathcal{Q}} + \mathcal{T}_{35,2}^{\mathcal{Q}} + \mathcal{T}_{35,3}^{\mathcal{Q}}, \end{aligned}$$

where $z = z_{12}^+$. Finally the sixth is a $\mathcal{R}'_{B,1}$ -type contribution,

$$\begin{aligned} \mathcal{T}_{36}^{\mathcal{Q}} &= \mathcal{A}_6(5, 6, \widehat{1}, 2, \ell, \widehat{P}) \frac{1}{(p_{5612} + \ell)^2} \mathcal{A}_4(-\widehat{P}, -\ell, 3, \widehat{4}) = \left(\frac{1}{p_{561}^2} + \frac{1}{p_{612}^2} + \frac{1}{(\ell + p_{12})^2} \right) \frac{1}{(p_{5612} + \ell)^2} \Big|_{\ell^2=0, z=-z_{34}^-} \\ &= \frac{1}{-2\ell \cdot p_{34} + p_{34}^2} \frac{1}{p_{561}^2} \Big|_{-z_{34}^-} + \frac{1}{-2\ell \cdot p_{34} + p_{34}^2} \frac{1}{p_{612}^2} \Big|_{-z_{34}^-} + \frac{1}{-2\ell \cdot p_{34} + p_{34}^2} \frac{1}{-2\ell \cdot p_{3456} + p_{3456}^2} \Big|_{-z_{34}^-} \\ &\equiv \mathcal{T}_{36,1}^{\mathcal{Q}} + \mathcal{T}_{36,2}^{\mathcal{Q}} + \mathcal{T}_{36,3}^{\mathcal{Q}}, \end{aligned} \tag{3.19}$$

where $z = -z_{34}^-$.

For tree diagrams of $\mathcal{T}_4^{\mathcal{Q}}$, there are in total five contributing terms. The first is a $\mathcal{R}_{A,2}^{\mathcal{Q}}$ -type contribution,

$$\mathcal{T}_{41}^{\mathcal{Q}} = \mathcal{A}_4(\widehat{P}, \widehat{1}, \widehat{\ell}_R, -\widehat{\ell}_L) \frac{1}{-2\ell \cdot p_{\widehat{1}\widehat{P}} + p_{\widehat{1}\widehat{P}}^2} \mathcal{A}_4(\widehat{\ell}_L, -\widehat{\ell}_R, 2, 3) \frac{1}{p_{123}^2} \mathcal{A}_4(-\widehat{P}, \widehat{4}, 5, 6) = \frac{1}{-2\ell \cdot p_{4561} + p_{4561}^2} \frac{1}{p_{123}^2},$$

where $z = z_{123}$, $\alpha = -\frac{p_{23}^2}{2\ell \cdot p_{23}}$. The second is a $\mathcal{R}_{A,1}^{\mathcal{Q}}$ -type contribution,

$$\mathcal{T}_{42}^{\mathcal{Q}} = \mathcal{A}_4(\widehat{1}, 2, 3, \widehat{P}) \frac{1}{p_{123}^2} \mathcal{A}_4(\widehat{4}, 5, \widehat{\ell}_R, -\widehat{\ell}_L) \frac{1}{-2\ell \cdot p_{45} + p_{45}^2} \mathcal{A}_4(\widehat{\ell}_L, -\widehat{\ell}_R, 6, -\widehat{P}) = \frac{1}{p_{123}^2} \frac{1}{-2\ell \cdot p_{45} + p_{45}^2} \Big|_{z_{123}},$$

where $z = z_{123}$, $\alpha = \frac{p_{45}^2}{2\ell \cdot p_{45}} \Big|_{z_{123}}$. The third is a $\mathcal{R}_{A,1}^{\mathcal{Q}}$ -type contribution,

$$\mathcal{T}_{43}^{\mathcal{Q}} = \mathcal{A}_4(6, \widehat{1}, 2, \widehat{P}) \frac{1}{p_{612}^2} \mathcal{A}_4(\widehat{4}, 5, \widehat{\ell}_R, -\widehat{\ell}_L) \frac{1}{-2\ell \cdot p_{45} + p_{45}^2} \mathcal{A}_4(\widehat{\ell}_L, -\widehat{\ell}_R, -\widehat{P}, 3) = \frac{1}{p_{612}^2} \frac{1}{-2\ell \cdot p_{45} + p_{45}^2} \Big|_{z_{612}},$$

where $z = z_{612}$, $\alpha = \frac{p_{45}^2}{2\ell \cdot p_{45}} \Big|_{z_{612}}$. The fourth is a $\mathcal{R}_{A,1}^Q$ -type contribution,

$$\mathcal{T}_{44}^Q = \mathcal{A}_4(5, 6, \hat{1}, \hat{P}) \frac{1}{p_{561}^2} \mathcal{A}_4(\hat{4}, -\hat{P}, \hat{\ell}_R, -\hat{\ell}_L) \frac{1}{-2\ell \cdot p_{4\hat{P}} + p_{4\hat{P}}^2} \mathcal{A}_4(\hat{\ell}_L, -\hat{\ell}_R, 2, 3) = \frac{1}{p_{561}^2} \frac{1}{-2\ell \cdot p_{4561} + p_{4561}^2},$$

where $z = z_{561}$, $\alpha = -\frac{p_{23}^2}{2\ell \cdot p_{23}}$. The fifth is a $\mathcal{R}'_{B,1}$ -type contribution,

$$\begin{aligned} \mathcal{T}_{45}^Q &= \mathcal{A}_6(6, \hat{1}, 2, 3, \ell, \hat{P}) \frac{1}{(p_{6123} + \ell)^2} \mathcal{A}_4(-\hat{P}, -\ell, \hat{4}, 5) = \left(\frac{1}{p_{612}^2} + \frac{1}{p_{123}^2} + \frac{1}{(\ell + p_{23})^2} \right) \frac{1}{(p_{6123} + \ell)^2} \Big|_{\ell^2=0, z=-z_{45}^-} \\ &= \frac{1}{-2\ell \cdot p_{45} + p_{45}^2} \frac{1}{p_{612}^2} \Big|_{-z_{45}^-} + \frac{1}{-2\ell \cdot p_{45} + p_{45}^2} \frac{1}{p_{123}^2} \Big|_{-z_{45}^-} + \frac{1}{-2\ell \cdot p_{45} + p_{45}^2} \frac{1}{-2\ell \cdot p_{4561} + p_{4561}^2} \\ &\equiv \mathcal{T}_{45,1}^Q + \mathcal{T}_{45,2}^Q + \mathcal{T}_{45,3}^Q, \end{aligned}$$

where $z = -z_{45}^-$.

For tree diagram of \mathcal{T}_5^Q , there are in total five contributing terms. The first is a $\mathcal{R}_{A,1}^Q$ -type contribution,

$$\mathcal{T}_{51}^Q = \mathcal{A}_4(\hat{1}, 2, 3, \hat{P}) \frac{1}{p_{123}^2} \mathcal{A}_4(5, 6, \hat{\ell}_R, -\hat{\ell}_L) \frac{1}{-2\ell \cdot p_{56} + p_{56}^2} \mathcal{A}_4(\hat{\ell}_L, -\hat{\ell}_R, -\hat{P}, \hat{4}) = \frac{1}{p_{123}^2} \frac{1}{-2\ell \cdot p_{56} + p_{56}^2},$$

where $z = z_{123}$, $\alpha = \frac{p_{56}^2}{2\ell \cdot p_{56}}$. The second is a $\mathcal{R}_{A,1}^Q$ -type contribution,

$$\mathcal{T}_{52}^Q = \mathcal{A}_4(6, \hat{1}, 2, \hat{P}) \frac{1}{p_{612}^2} \mathcal{A}_4(5, -\hat{P}, \hat{\ell}_R, -\hat{\ell}_L) \frac{1}{-2\ell \cdot p_{5\hat{P}} + p_{5\hat{P}}^2} \mathcal{A}_4(\hat{\ell}_L, -\hat{\ell}_R, 3, \hat{4}) = \frac{1}{p_{612}^2} \frac{1}{-2\ell \cdot p_{56\hat{1}2} + p_{56\hat{1}2}^2} \Big|_{z_{612}},$$

where $z = z_{612}$, $\alpha = -\frac{p_{34}^2}{2\ell \cdot p_{34}} \Big|_{z_{612}}$. The third is a $\mathcal{R}_{A,1}^Q$ -type contribution,

$$\mathcal{T}_{53}^Q = \mathcal{A}_4(5, 6, \hat{1}, \hat{P}) \frac{1}{p_{561}^2} \mathcal{A}_4(-\hat{P}, 2, \hat{\ell}_R, -\hat{\ell}_L) \frac{1}{-2\ell \cdot p_{2\hat{P}} + p_{2\hat{P}}^2} \mathcal{A}_4(\hat{\ell}_L, -\hat{\ell}_R, 3, \hat{4}) = \frac{1}{p_{561}^2} \frac{1}{-2\ell \cdot p_{56\hat{1}2} + p_{56\hat{1}2}^2} \Big|_{z_{561}},$$

where $z = z_{561}$, $\alpha = -\frac{p_{34}^2}{2\ell \cdot p_{34}} \Big|_{z_{561}}$. The fourth is a $\mathcal{R}_{A,2}^Q$ -type contribution,

$$\mathcal{T}_{54}^Q = \mathcal{A}_4(5, 6, \hat{\ell}_R, -\hat{\ell}_L) \frac{1}{-2\ell \cdot p_{56} + p_{56}^2} \mathcal{A}_4(\hat{\ell}_L, -\hat{\ell}_R, \hat{1}, \hat{P}) \frac{1}{p_{561}^2} \mathcal{A}_4(-\hat{P}, 2, 3, \hat{4}) = \frac{1}{-2\ell \cdot p_{56} + p_{56}^2} \frac{1}{p_{561}^2},$$

where $z = z_{561}$, $\alpha = \frac{p_{56}^2}{2\ell \cdot p_{56}}$. The fifth is a $\mathcal{R}'_{B,2}$ -type contribution,

$$\begin{aligned} \mathcal{T}_{55}^Q &= \mathcal{A}_6(-\ell, 5, 6, \hat{1}, 2, \hat{P}) \frac{1}{(p_{5612} - \ell)^2} \mathcal{A}_4(-\hat{P}, 3, \hat{4}, \ell) = \left(\frac{1}{p_{561}^2} + \frac{1}{p_{612}^2} + \frac{1}{(p_{56} - \ell)^2} \right) \frac{1}{(p_{5612} - \ell)^2} \Big|_{\ell^2=0, z=-z_{34}^+} \\ &= \frac{1}{-2\ell \cdot p_{5612} + p_{5612}^2} \frac{1}{p_{561}^2} \Big|_{-z_{34}^+} + \frac{1}{-2\ell \cdot p_{5612} + p_{5612}^2} \frac{1}{p_{612}^2} \Big|_{-z_{34}^+} + \frac{1}{-2\ell \cdot p_{5612} + p_{5612}^2} \frac{1}{-2\ell \cdot p_{56} + p_{56}^2} \\ &\equiv \mathcal{T}_{55,1}^Q + \mathcal{T}_{55,2}^Q + \mathcal{T}_{55,3}^Q, \end{aligned}$$

where $z = -z_{34}^+$.

For tree diagrams of \mathcal{T}_6^Q , there are in total six contributing terms. The first is a $\mathcal{R}_{A,1}^Q$ -type contribution,

$$\mathcal{T}_{61}^Q = \mathcal{A}_4(\hat{1}, 2, 3, \hat{P}) \frac{1}{p_{123}^2} \mathcal{A}_4(6, -\hat{P}, \hat{\ell}_R, -\hat{\ell}_L) \frac{1}{-2\ell \cdot p_{6\hat{P}} + p_{6\hat{P}}^2} \mathcal{A}_4(\hat{\ell}_L, -\hat{\ell}_R, \hat{4}, 5) = \frac{1}{p_{123}^2} \frac{1}{-2\ell \cdot p_{6123} + p_{6123}^2} \Big|_{z_{123}},$$

where $z = z_{123}$, $\alpha = -\frac{p_{45}^2}{2\ell \cdot p_{45}} \Big|_{z_{123}}$. The second is a $\mathcal{R}_{A,1}^Q$ -type contribution,

$$\mathcal{T}_{62}^Q = \mathcal{A}_4(6, \hat{1}, 2, \hat{P}) \frac{1}{p_{612}^2} \mathcal{A}_4(-\hat{P}, 3, \hat{\ell}_R, -\hat{\ell}_L) \frac{1}{-2\ell \cdot p_{3\hat{P}} + p_{3\hat{P}}^2} \mathcal{A}_4(\hat{\ell}_L, -\hat{\ell}_R, \hat{4}, 5) = \frac{1}{p_{612}^2} \frac{1}{-2\ell \cdot p_{6\hat{1}23} + p_{6\hat{1}23}^2} \Big|_{z_{612}},$$

where $z = z_{612}$, $\alpha = -\frac{p_{45}^2}{2\ell \cdot p_{45}} \Big|_{z_{612}}$. The third is a $\mathcal{R}_{A,2}^Q$ -type contribution,

$$\mathcal{T}_{63}^Q = \mathcal{A}_4(6, \hat{1}, \hat{\ell}_R, -\hat{\ell}_L) \frac{1}{-2\ell \cdot p_{6\hat{1}} + p_{6\hat{1}}^2} \mathcal{A}_4(\hat{\ell}_L, -\hat{\ell}_R, 2, \hat{P}) \frac{1}{p_{612}^2} \mathcal{A}_4(-\hat{P}, 3, \hat{4}, 5) = \frac{1}{-2\ell \cdot p_{6\hat{1}} + p_{6\hat{1}}^2} \frac{1}{p_{612}^2} \Big|_{z_{612}},$$

where $z = z_{612}$, $\alpha = \frac{p_{6\hat{1}}^2}{2\ell \cdot p_{6\hat{1}}} \Big|_{z_{612}}$. The fourth is a $\mathcal{R}_{A,2}^Q$ -type contribution,

$$\mathcal{T}_{64}^Q = \mathcal{A}_4(6, \hat{1}, \hat{\ell}_R, -\hat{\ell}_L) \frac{1}{-2\ell \cdot p_{6\hat{1}} + p_{6\hat{1}}^2} \mathcal{A}_4(\hat{\ell}_L, -\hat{\ell}_R, \hat{P}, 5) \frac{1}{p_{561}^2} \mathcal{A}_4(-\hat{P}, 2, 3, \hat{4}) = \frac{1}{-2\ell \cdot p_{6\hat{1}} + p_{6\hat{1}}^2} \frac{1}{p_{561}^2} \Big|_{z_{561}},$$

where $z = z_{561}$, $\alpha = \frac{p_{6\hat{1}}^2}{2\ell \cdot p_{6\hat{1}}} \Big|_{z_{561}}$. The fifth is a $\mathcal{R}'_{B,2}$ -type contribution,

$$\begin{aligned} \mathcal{T}_{65}^Q &= \mathcal{A}_4(-\ell, 6, \hat{1}, \hat{P}) \frac{1}{(p_{61} - \ell)^2} \mathcal{A}_6(-\hat{P}, 2, 3, \hat{4}, 5, \ell) = \frac{1}{(p_{61} - \ell)^2} \left(\frac{1}{p_{234}^2} + \frac{1}{p_{345}^2} + \frac{1}{(\ell + p_{45})^2} \right) \Big|_{\ell^2=0, z=z_{61}^-} \\ &= \frac{1}{-2\ell \cdot p_{61} + p_{61}^2} \frac{1}{p_{234}^2} \Big|_{z_{61}^-} + \frac{1}{-2\ell \cdot p_{61} + p_{61}^2} \frac{1}{p_{345}^2} \Big|_{z_{61}^-} + \frac{1}{-2\ell \cdot p_{61} + p_{61}^2} \frac{1}{-2\ell \cdot p_{6\hat{1}23} + p_{6\hat{1}23}^2} \Big|_{z_{61}^-} \\ &\equiv \mathcal{T}_{65,1}^Q + \mathcal{T}_{65,2}^Q + \mathcal{T}_{65,3}^Q, \end{aligned}$$

where $z = z_{61}^-$. Finally, the sixth is a $\mathcal{R}'_{B,2}$ -type contribution,

$$\begin{aligned} \mathcal{T}_{66}^Q &= \mathcal{A}_6(-\ell, 6, \hat{1}, 2, 3, \hat{P}) \frac{1}{(p_{6123} - \ell)^2} \mathcal{A}_4(-\hat{P}, \hat{4}, 5, \ell) = \left(\frac{1}{p_{612}^2} + \frac{1}{p_{123}^2} + \frac{1}{(-\ell + p_{6\hat{1}})^2} \right) \frac{1}{(p_{6123} - \ell)^2} \Big|_{\ell^2=0, z=-z_{45}^+} \\ &= \frac{1}{-2\ell \cdot p_{6123} + p_{6123}^2} \frac{1}{p_{612}^2} \Big|_{-z_{45}^+} + \frac{1}{-2\ell \cdot p_{6123} + p_{6123}^2} \frac{1}{p_{123}^2} \Big|_{-z_{45}^+} + \frac{1}{-2\ell \cdot p_{6123} + p_{6123}^2} \frac{1}{-2\ell \cdot p_{6\hat{1}} + p_{6\hat{1}}^2} \Big|_{-z_{45}^+} \\ &\equiv \mathcal{T}_{66,1}^Q + \mathcal{T}_{66,2}^Q + \mathcal{T}_{66,3}^Q, \end{aligned}$$

where $z = -z_{45}^+$.

All the above results in total generate 48 terms. As is done for \mathcal{T}_1^Q , it can be checked that, the 4 terms with un-deformed momenta

$$\begin{aligned} &\frac{1}{\ell^2} (\mathcal{T}_{15,3}^Q + \mathcal{T}_{25,3}^Q + \mathcal{T}_{45,3}^Q + \mathcal{T}_{55,3}^Q) \\ &= \frac{1}{\ell^2 (-2\ell \cdot p_{12} + p_{12}^2) (-2\ell \cdot p_{1234} + p_{1234}^2)} + \frac{1}{\ell^2 (-2\ell \cdot p_{23} + p_{23}^2) (-2\ell \cdot p_{2345} + p_{2345}^2)} \\ &\quad + \frac{1}{\ell^2 (-2\ell \cdot p_{45} + p_{45}^2) (-2\ell \cdot p_{4561} + p_{4561}^2)} + \frac{1}{\ell^2 (-2\ell \cdot p_{56} + p_{56}^2) (-2\ell \cdot p_{5612} + p_{5612}^2)} \end{aligned} \tag{3.20}$$

reproduce the 4 four terms in the first line of (3.3). While

$$\frac{1}{\ell^2} (\mathcal{T}_{35,3}^Q + \mathcal{T}_{36,3}^Q) = \frac{1}{\ell^2 (-2\ell \cdot p_{34} + p_{34}^2) (-2\ell \cdot p_{3456} + p_{3456}^2)} \tag{3.21}$$

and

$$\frac{1}{\ell^2}(\mathcal{T}_{65,3}^{\mathcal{Q}} + \mathcal{T}_{66,3}^{\mathcal{Q}}) = \frac{1}{\ell^2(-2\ell \cdot p_{61} + p_{61}^2)(-2\ell \cdot p_{6123} + p_{6123}^2)} \quad (3.22)$$

reproduce the other 2 in the first line of (3.3).

For the comparison of the second line in (3.3), we have

$$\begin{aligned} & \frac{1}{\ell^2}(\mathcal{T}_{21}^{\mathcal{Q}} + \mathcal{T}_{24}^{\mathcal{Q}} + \mathcal{T}_{41}^{\mathcal{Q}} + \mathcal{T}_{44}^{\mathcal{Q}} + \mathcal{T}_{51}^{\mathcal{Q}} + \mathcal{T}_{54}^{\mathcal{Q}} + \mathcal{T}_{12}^{\mathcal{Q}} + \mathcal{T}_{14}^{\mathcal{Q}}) \\ &= \left(\frac{1}{\ell^2(-2\ell \cdot p_{23} + p_{23}^2)} + \frac{1}{\ell^2(-2\ell \cdot p_{4561} + p_{4561}^2)} \right) \left(\frac{1}{p_{456}^2} + \frac{1}{p_{561}^2} \right) \\ &+ \left(\frac{1}{\ell^2(-2\ell \cdot p_{56} + p_{56}^2)} + \frac{1}{\ell^2(-2\ell \cdot p_{1234} + p_{1234}^2)} \right) \left(\frac{1}{p_{123}^2} + \frac{1}{p_{234}^2} \right), \end{aligned} \quad (3.23)$$

as well as

$$\begin{aligned} & \frac{1}{\ell^2}(\mathcal{T}_{11}^{\mathcal{Q}} + \mathcal{T}_{15,2}^{\mathcal{Q}} + \mathcal{T}_{13}^{\mathcal{Q}} + \mathcal{T}_{15,1}^{\mathcal{Q}}) + \frac{1}{\ell^2}(\mathcal{T}_{31}^{\mathcal{Q}} + \mathcal{T}_{35,2}^{\mathcal{Q}} + \mathcal{T}_{32}^{\mathcal{Q}} + \mathcal{T}_{35,1}^{\mathcal{Q}}) \\ &= \frac{1}{\ell^2(-2\ell \cdot p_{12} + p_{12}^2)} \left(\frac{1}{p_{345}^2} + \frac{1}{p_{456}^2} \right) + \frac{1}{\ell^2(-2\ell \cdot p_{3456} + p_{3456}^2)} \left(\frac{1}{p_{345}^2} + \frac{1}{p_{456}^2} \right), \end{aligned} \quad (3.24)$$

and

$$\begin{aligned} & \frac{1}{\ell^2}(\mathcal{T}_{33}^{\mathcal{Q}} + \mathcal{T}_{36,2}^{\mathcal{Q}} + \mathcal{T}_{34}^{\mathcal{Q}} + \mathcal{T}_{36,1}^{\mathcal{Q}}) + \frac{1}{\ell^2}(\mathcal{T}_{52}^{\mathcal{Q}} + \mathcal{T}_{55,2}^{\mathcal{Q}} + \mathcal{T}_{53}^{\mathcal{Q}} + \mathcal{T}_{55,1}^{\mathcal{Q}}) \\ &= \frac{1}{\ell^2(-2\ell \cdot p_{34} + p_{34}^2)} \left(\frac{1}{p_{561}^2} + \frac{1}{p_{612}^2} \right) + \frac{1}{\ell^2(-2\ell \cdot p_{5612} + p_{5612}^2)} \left(\frac{1}{p_{561}^2} + \frac{1}{p_{612}^2} \right), \end{aligned} \quad (3.25)$$

and

$$\begin{aligned} & \frac{1}{\ell^2}(\mathcal{T}_{42}^{\mathcal{Q}} + \mathcal{T}_{45,2}^{\mathcal{Q}} + \mathcal{T}_{43}^{\mathcal{Q}} + \mathcal{T}_{45,1}^{\mathcal{Q}}) + \frac{1}{\ell^2}(\mathcal{T}_{61}^{\mathcal{Q}} + \mathcal{T}_{66,2}^{\mathcal{Q}} + \mathcal{T}_{62}^{\mathcal{Q}} + \mathcal{T}_{66,1}^{\mathcal{Q}}) \\ &= \frac{1}{\ell^2(-2\ell \cdot p_{45} + p_{45}^2)} \left(\frac{1}{p_{612}^2} + \frac{1}{p_{123}^2} \right) + \frac{1}{\ell^2(-2\ell \cdot p_{6123} + p_{6123}^2)} \left(\frac{1}{p_{612}^2} + \frac{1}{p_{123}^2} \right), \end{aligned} \quad (3.26)$$

and

$$\begin{aligned} & \frac{1}{\ell^2}(\mathcal{T}_{63}^{\mathcal{Q}} + \mathcal{T}_{65,2}^{\mathcal{Q}} + \mathcal{T}_{64}^{\mathcal{Q}} + \mathcal{T}_{65,1}^{\mathcal{Q}}) + \frac{1}{\ell^2}(\mathcal{T}_{22}^{\mathcal{Q}} + \mathcal{T}_{25,2}^{\mathcal{Q}} + \mathcal{T}_{23}^{\mathcal{Q}} + \mathcal{T}_{25,1}^{\mathcal{Q}}) \\ &= \frac{1}{\ell^2(-2\ell \cdot p_{61} + p_{61}^2)} \left(\frac{1}{p_{234}^2} + \frac{1}{p_{345}^2} \right) + \frac{1}{\ell^2(-2\ell \cdot p_{2345} + p_{2345}^2)} \left(\frac{1}{p_{234}^2} + \frac{1}{p_{345}^2} \right). \end{aligned} \quad (3.27)$$

Thus we confirm the equivalence among results of Feynman diagram method, \mathcal{Q} -cut representation and recursive formula (2.33) term by term. In fact, by cyclic invariance, we can rewrite the integrand (3.3) as

$$\begin{aligned} \mathcal{I}^{\mathcal{F}} = & \left\{ \frac{1}{\ell^2(-2\ell \cdot p_{12} + p_{12}^2)(-2\ell \cdot p_{1234} + p_{1234}^2)} + \frac{1}{\ell^2(-2\ell \cdot p_{12} + p_{12}^2)} \left(\frac{1}{p_{345}^2} + \frac{1}{p_{456}^2} \right) \right. \\ & \left. + \frac{1}{\ell^2(-2\ell \cdot p_{1234} + p_{1234}^2)} \left(\frac{1}{p_{123}^2} + \frac{1}{p_{234}^2} \right) \right\} + \text{Cyclic}\{1, 2, 3, 4, 5, 6\}. \end{aligned} \quad (3.28)$$

Then the recursive formula of tree diagram $\mathcal{T}^{\mathcal{Q}}(\ell, -\ell, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)$ reproduces the result under the same ordering in $\mathcal{I}^{\mathcal{F}}$. For instance, result of $\mathcal{T}^{\mathcal{Q}}(\ell, -\ell, 1, 2, 3, 4, 5, 6)$ reproduces the above result in the curly bracket.

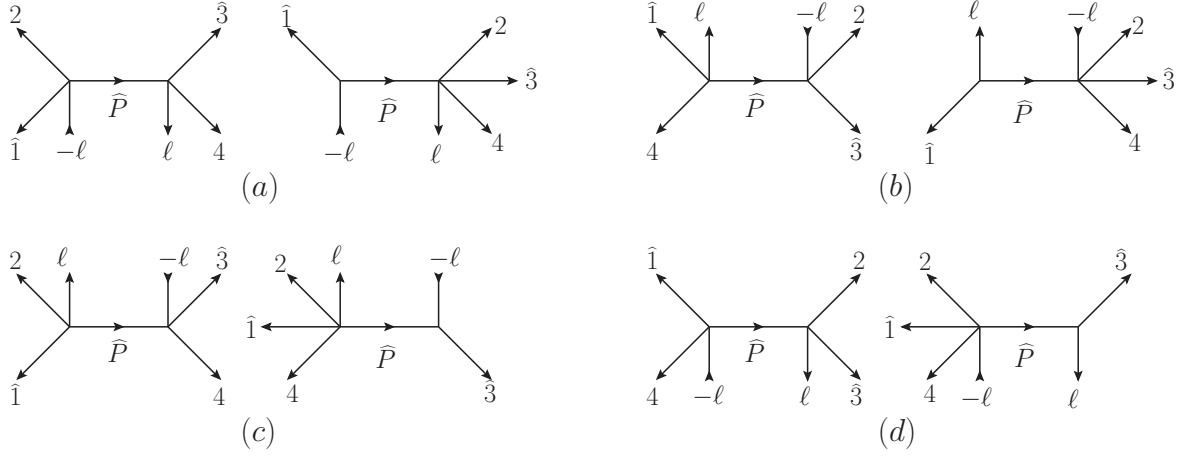


Figure 2. Non-vanishing diagrams for (a) $\mathcal{T}_1(\ell, -\ell, 1, 2, 3, 4)$, (b) $\mathcal{T}_2(\ell, -\ell, 2, 3, 4, 1)$, (c) $\mathcal{T}_3(\ell, -\ell, 3, 4, 1, 2)$, (d) $\mathcal{T}_4(\ell, -\ell, 4, 1, 2, 3)$, under p_1, p_3 BCFW deformation.

3.2 The one-loop four-point amplitude in scalar ϕ^3 theory

Let us now discuss the integrand of one-loop four-point amplitude in color-ordered scalar ϕ^3 theory, so the tree diagram $\mathcal{T}^{\mathcal{Q}}$ have four contributions, denoted as

$$\mathcal{T}^{\mathcal{Q}} = \mathcal{T}_1^{\mathcal{Q}}(\ell, -\ell, 1, 2, 3, 4) + \mathcal{T}_2^{\mathcal{Q}}(\ell, -\ell, 2, 3, 4, 1) + \mathcal{T}_3^{\mathcal{Q}}(\ell, -\ell, 3, 4, 1, 2) + \mathcal{T}_4^{\mathcal{Q}}(\ell, -\ell, 4, 1, 2, 3) . \quad (3.29)$$

The momentum deformation is taken as

$$\hat{p}_1 = p_1 + zq \quad , \quad \hat{p}_3 = p_3 - zq \quad , \quad q \cdot p_{1,3} = q^2 = 0 .$$

Under the given momentum deformation, each $\mathcal{T}_i^{\mathcal{Q}}$ has two non-vanishing terms⁵, as shown in Figure 2. Recall that the integrand of one-loop four-point amplitude in scalar ϕ^3 theory, after partial fraction identity, is given by [2]

$$\mathcal{I}^{\mathcal{F}}(1, 2, 3, 4) = \frac{1}{\ell^2} \left(\frac{1}{-2\ell \cdot p_1} + \frac{1}{p_{12}^2} \right) \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \left(\frac{1}{2\ell \cdot p_4} + \frac{1}{p_{34}^2} \right) + \text{Cyclic}\{1, 2, 3, 4\} . \quad (3.30)$$

We want to show that, the integrand given by recursive formula (2.33) is equivalent to (3.30), up to certain scale free terms.

Let us start by computing the two diagrams in Figure 2.a. The four-point tree amplitude is

$$\mathcal{A}_4(1, 2, 3, 4) = \frac{1}{(p_1 + p_2)^2} + \frac{1}{(p_2 + p_3)^2} , \quad (3.31)$$

⁵Since the one-loop integrand $\mathcal{I}_2^{\mathcal{Q}} = \mathcal{I}_3^{\mathcal{Q}} = 0$, the contributions to $\mathcal{R}_{A,1}^{\mathcal{Q}}, \mathcal{R}_{A,2}^{\mathcal{Q}}$ will be zero. However, all $\mathcal{R}'_{B,1}, \mathcal{R}''_{B,1}, \mathcal{R}'''_{B,1}, \mathcal{R}'_{B,2}, \mathcal{R}''_{B,2}, \mathcal{R}'''_{B,2}$ will contribute.

and let us define

$$z_{12}^{\pm} \equiv -\frac{\pm 2\ell \cdot p_{12} + p_{12}^2}{2q \cdot (p_{12} \pm \ell)} \quad , \quad z_{41}^{\pm} \equiv -\frac{\pm 2\ell \cdot p_{41} + p_{41}^2}{2q \cdot (p_{41} \pm \ell)} \quad , \quad z_1 \equiv -\frac{2\ell \cdot p_1}{2q \cdot \ell} \quad , \quad z_3 \equiv \frac{2\ell \cdot p_3}{2q \cdot \ell} . \quad (3.32)$$

The first diagram gives a $\mathcal{R}'_{B,2}$ -type contribution,

$$\begin{aligned} \mathcal{T}_{11}^{\mathcal{Q}} &= \mathcal{A}_4(-\ell, \hat{1}, 2, \hat{P}) \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \mathcal{A}_4(-\hat{P}, \hat{3}, 4, \ell) \\ &= \left(\frac{1}{-2\ell \cdot \hat{p}_1} + \frac{1}{\hat{p}_{12}^2} \right) \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \left(\frac{1}{2\ell \cdot p_4} + \frac{1}{\hat{p}_{34}^2} \right) \Big|_{z_{12}^-} \equiv \mathcal{T}_{11,1}^{\mathcal{Q}} + \mathcal{T}_{11,2}^{\mathcal{Q}} + \mathcal{T}_{11,3}^{\mathcal{Q}} + \mathcal{T}_{11,4}^{\mathcal{Q}} , \end{aligned} \quad (3.33)$$

where $z = z_{12}^-$, and $\mathcal{T}_{11,i}^{\mathcal{Q}}$ denotes the four terms after expanding the result. The second diagram gives a $\mathcal{R}''_{B,2}$ -type contribution,

$$\begin{aligned} \mathcal{T}_{12}^{\mathcal{Q}} &= \mathcal{A}_3(-\ell, \hat{1}, \hat{P}) \frac{1}{-2\ell \cdot p_1} \mathcal{A}_3(-\hat{P}, 2, P') \frac{1}{2\ell \cdot \hat{p}_{34} + \hat{p}_{34}^2} \mathcal{A}_4(-P', \hat{3}, 4, \alpha\ell) \\ &= \frac{1}{-2\ell \cdot p_1} \frac{1}{-2\ell \cdot \hat{p}_{12} + \hat{p}_{12}^2} \left(\frac{1}{\alpha(2\ell \cdot p_4)} + \frac{1}{\hat{p}_{34}^2} \right) \Big|_{z=z_1, \alpha=\alpha_{12}} , \end{aligned} \quad (3.34)$$

where \hat{P}, P' are understood to follow the momentum conservation of each sub-amplitudes, and

$$\alpha_{12} = -\frac{\hat{p}_{34}^2}{2\ell \cdot \hat{p}_{34}} = \frac{\hat{p}_{12}^2}{2\ell \cdot \hat{p}_{12}} \Big|_{z=z_1} . \quad (3.35)$$

In fact, when substituting α_{12} back in $\mathcal{T}_{12}^{\mathcal{Q}}$, we get

$$\begin{aligned} \mathcal{T}_{12}^{\mathcal{Q}} &= \frac{1}{-2\ell \cdot p_1} \frac{1}{-2\ell \cdot \hat{p}_{12} + \hat{p}_{12}^2} \left(-\frac{-2\ell \cdot \hat{p}_{12} + \hat{p}_{12}^2}{\hat{p}_{12}^2(2\ell \cdot p_4)} + \frac{1}{2\ell \cdot p_4} + \frac{1}{\hat{p}_{34}^2} \right) \Big|_{z=z_1} \\ &= -\frac{1}{-2\ell \cdot p_1} \frac{1}{\hat{p}_{12}^2(2\ell \cdot p_4)} \Big|_{z=z_1} + \frac{1}{-2\ell \cdot p_1} \frac{1}{-2\ell \cdot \hat{p}_{12} + \hat{p}_{12}^2} \left(\frac{1}{2\ell \cdot p_4} + \frac{1}{\hat{p}_{34}^2} \right) \Big|_{z=z_1} . \end{aligned} \quad (3.36)$$

Note that

$$\hat{p}_{12}^2|_{z_1} = p_{12}^2 + z_1(2q \cdot p_{12}) = \frac{2K_1 \cdot \ell}{2q \cdot \ell} \quad , \quad K_1 \equiv (p_{12}^2)q - (2q \cdot p_{12})p_1 , \quad (3.37)$$

so the first term in (3.36) is a scale free term and can be ignored. Hence we have four terms from $\mathcal{T}_{11}^{\mathcal{Q}}$ and two terms from $\mathcal{T}_{12}^{\mathcal{Q}}$, and we want to compare the sum $\frac{1}{\ell^2}(\mathcal{T}_{11,1}^{\mathcal{Q}} + \mathcal{T}_{11,2}^{\mathcal{Q}} + \mathcal{T}_{11,3}^{\mathcal{Q}} + \mathcal{T}_{11,4}^{\mathcal{Q}} + \mathcal{T}_{12,1}^{\mathcal{Q}} + \mathcal{T}_{12,2}^{\mathcal{Q}})$ with

$$\frac{1}{\ell^2} \left(\frac{1}{-2\ell \cdot p_1} + \frac{1}{p_{12}^2} \right) \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \left(\frac{1}{2\ell \cdot p_4} + \frac{1}{p_{34}^2} \right) \equiv \mathcal{I}_{1,1}^{\mathcal{F}} + \mathcal{I}_{1,2}^{\mathcal{F}} + \mathcal{I}_{1,3}^{\mathcal{F}} + \mathcal{I}_{1,4}^{\mathcal{F}} . \quad (3.38)$$

To see the correspondence explicitly, firstly we have

$$\begin{aligned} &\mathcal{T}_{11,1}^{\mathcal{Q}} + \mathcal{T}_{12,1}^{\mathcal{Q}} \\ &= \frac{1}{-2\ell \cdot \hat{p}_1} \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \frac{1}{2\ell \cdot p_4} \Big|_{z_{12}^-} + \frac{1}{-2\ell \cdot p_1} \frac{1}{-2\ell \cdot \hat{p}_{12} + \hat{p}_{12}^2} \frac{1}{2\ell \cdot p_4} \Big|_{z_1} \\ &= \left(\frac{1}{(-2\ell \cdot p_1) + \lambda(-2\ell \cdot p_{12} + p_{12}^2)} \frac{1}{(-2\ell \cdot p_{12} + p_{12}^2)} + \frac{1}{(-2\ell \cdot p_1)} \frac{1}{(-2\ell \cdot p_{12} + p_{12}^2) + (-2\ell \cdot p_1)/\lambda} \right) \frac{1}{2\ell \cdot p_4} \\ &= \frac{1}{-2\ell \cdot p_1} \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \frac{1}{2\ell \cdot p_4} , \end{aligned} \quad (3.39)$$

where $\lambda = \frac{2q \cdot \ell}{2q \cdot (p_{12} - \ell)}$, and in the last line we have used the identity (3.17). So we see that

$$\frac{1}{\ell^2}(\mathcal{T}_{11,1}^{\mathcal{Q}} + \mathcal{T}_{12,1}^{\mathcal{Q}}) = \mathcal{I}_{1,1}^{\mathcal{F}}. \quad (3.40)$$

Next, we have

$$\begin{aligned} \frac{1}{\ell^2} \mathcal{T}_{11,3}^{\mathcal{Q}} - \mathcal{I}_{1,3}^{\mathcal{F}} &= \frac{1}{\ell^2} \left(\frac{1}{\widehat{p}_{12}^2} \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \frac{1}{2\ell \cdot p_4} \Big|_{z_{12}^-} - \frac{1}{p_{12}^2} \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \frac{1}{2\ell \cdot p_4} \right) \\ &= -\frac{2q \cdot p_{12}}{\ell^2(2K_{12} \cdot \ell)(2\ell \cdot p_4)p_{12}^2}, \quad K_{12} \equiv (p_{12}^2)q - (2q \cdot p_{12})p_{12}. \end{aligned} \quad (3.41)$$

So $\frac{1}{\ell^2} \mathcal{T}_{11,3}^{\mathcal{Q}}$ is equivalent to $\mathcal{I}_{1,3}^{\mathcal{F}}$, up to a scale free term. Similarly,

$$\begin{aligned} \frac{1}{\ell^2} \mathcal{T}_{11,4}^{\mathcal{Q}} - \mathcal{I}_{1,4}^{\mathcal{F}} &= \frac{1}{\ell^2} \left(\frac{1}{\widehat{p}_{12}^2} \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \frac{1}{\widehat{p}_{34}^2} \Big|_{z_{12}^-} - \frac{1}{p_{12}^2} \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \frac{1}{p_{34}^2} \right) \\ &= -\frac{(2q \cdot p_{12})(2q \cdot \ell)}{\ell^2(2K_{12} \cdot \ell)^2 p_{12}^2} + \frac{(2q \cdot p_{12})^2}{\ell^2(2K_{12} \cdot \ell)^2 p_{12}^2} - \frac{2q \cdot p_{12}}{\ell^2(2K_{12} \cdot \ell)(p_{12}^2)^2}, \end{aligned} \quad (3.42)$$

which is also a scale free term.

Finally, we have

$$\begin{aligned} &\frac{1}{\ell^2}(\mathcal{T}_{11,2}^{\mathcal{Q}} + \mathcal{T}_{12,2}^{\mathcal{Q}}) - \mathcal{I}_{1,2}^{\mathcal{F}} \\ &= \frac{1}{\ell^2} \left(\frac{1}{-2\ell \cdot \widehat{p}_1} \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \frac{1}{\widehat{p}_{34}^2} \Big|_{z_{12}^-} + \frac{1}{-2\ell \cdot p_1} \frac{1}{-2\ell \cdot \widehat{p}_{12} + \widehat{p}_{12}^2} \frac{1}{\widehat{p}_{34}^2} \Big|_{z_1} - \frac{1}{-2\ell \cdot p_1} \frac{1}{-2\ell \cdot p_{12} + p_{12}^2} \frac{1}{p_{34}^2} \right) \\ &= \frac{1}{\ell^2} \left(\frac{p_{12}^2(2q \cdot p_{12} - 2q \cdot \ell)^2(2\ell \cdot p_1)F_3 + p_{12}^2(2q \cdot \ell)^2(-2\ell \cdot p_{12} + p_{12}^2)F_2 - F_1 F_2 F_3}{p_{12}^2(-2\ell \cdot p_1)(-2\ell \cdot p_{12} + p_{12}^2)F_1 F_2 F_3} \right) \\ &= -\frac{(2q \cdot p_{12})^2}{\ell^2(2K_1 \cdot \ell)(2K_{12} \cdot \ell)p_{12}^2}, \end{aligned} \quad (3.43)$$

where

$$F_1 \equiv p_{12}^2(2q \cdot \ell) - (2\ell \cdot p_1)(2q \cdot p_{12}) - (2\ell \cdot p_2)(2q \cdot \ell), \quad (3.44)$$

$$F_2 \equiv 2K_{12} \cdot \ell = p_{12}^2(2q \cdot \ell) - (2\ell \cdot p_1)(2q \cdot p_{12}) - (2\ell \cdot p_2)(2q \cdot p_{12}), \quad (3.45)$$

$$F_3 \equiv 2K_1 \cdot \ell = p_{12}^2(2q \cdot \ell) - (2\ell \cdot p_1)(2q \cdot p_{12}). \quad (3.46)$$

Thus we conclude that

$$\begin{aligned} \frac{1}{\ell^2} \mathcal{T}_1^{\mathcal{Q}} &= \mathcal{I}_1^{\mathcal{F}} + \frac{2q \cdot \ell}{\ell^2(-2\ell \cdot p_1)(2K_1 \cdot \ell)(2\ell \cdot p_4)} - \frac{2q \cdot p_{12}}{\ell^2(2K_{12} \cdot \ell)(2\ell \cdot p_4)p_{12}^2} - \frac{(2q \cdot p_{12})(2q \cdot \ell)}{\ell^2(2K_{12} \cdot \ell)^2 p_{12}^2} \\ &\quad + \frac{(2q \cdot p_{12})^2}{\ell^2(2K_{12} \cdot \ell)^2 p_{12}^2} - \frac{2q \cdot p_{12}}{\ell^2(2K_{12} \cdot \ell)(p_{12}^2)^2} - \frac{(2q \cdot p_{12})^2}{\ell^2(2K_1 \cdot \ell)(2K_{12} \cdot \ell)p_{12}^2}. \end{aligned}$$

It confirms that, the result of recursive formula (2.33) is equivalent to the result of Feynman diagram method, up to some scale free terms.

The same computation can be applied to tree diagrams $\mathcal{T}_2^Q, \mathcal{T}_3^Q$ and \mathcal{T}_4^Q . For \mathcal{T}_2^Q , we have two contributing diagrams as shown in Figure 2.b, and we get

$$\begin{aligned}\mathcal{T}_{21}^Q &= \mathcal{A}_4(4, \hat{1}, \ell, \hat{P}) \frac{1}{2\ell \cdot p_{41} + p_{41}^2} \mathcal{A}_4(-\hat{P}, -\ell, 2, \hat{3}) \\ &= \left(\frac{1}{2\ell \cdot \hat{p}_1} + \frac{1}{\hat{p}_{41}^2} \right) \frac{1}{-2\ell \cdot p_{23} + p_{23}^2} \left(\frac{1}{-2\ell \cdot p_2} + \frac{1}{\hat{p}_{23}^2} \right) \Big|_{z_{41}^+} \equiv \mathcal{T}_{21,1}^Q + \mathcal{T}_{21,2}^Q + \mathcal{T}_{21,3}^Q + \mathcal{T}_{21,4}^Q, \quad (3.47)\end{aligned}$$

as well as

$$\begin{aligned}\mathcal{T}_{22}^Q &= \mathcal{A}_3(\hat{1}, \ell, \hat{P}) \frac{1}{2\ell \cdot p_1} \mathcal{A}_4(-\alpha\ell, 2, \hat{3}, P') \frac{1}{-2\ell \cdot \hat{p}_{23} + \hat{p}_{23}^2} \mathcal{A}_3(-P', 4, -\hat{P}) \\ &= \frac{2q \cdot \ell}{(2K'_1 \cdot \ell)(2\ell \cdot p_1)(2\ell \cdot p_2)} + \frac{1}{2\ell \cdot p_1} \left(\frac{1}{-2\ell \cdot p_2} + \frac{1}{\hat{p}_{23}^2} \right) \frac{1}{-2\ell \cdot \hat{p}_{23} + \hat{p}_{23}^2} \Big|_{z_1}, \quad (3.48)\end{aligned}$$

where $K'_1 \equiv (p_{23}^2)q + (2q \cdot p_{23})p_1$,

$$\alpha_{23} = \frac{\hat{p}_{23}^2}{2\ell \cdot \hat{p}_{23}} \Big|_{z=z_1}.$$

The first term in (3.48) is scale free, while the second and third terms are denoted as $\mathcal{T}_{22,1}^Q, \mathcal{T}_{22,2}^Q$. The result $\frac{1}{\ell^2} \mathcal{T}_2^Q$ is equivalent to

$$\frac{1}{\ell^2} \left(\frac{1}{-2\ell \cdot p_2} + \frac{1}{p_{23}^2} \right) \frac{1}{-2\ell \cdot p_{23} + p_{23}^2} \left(\frac{1}{2\ell \cdot p_1} + \frac{1}{p_{41}^2} \right) \equiv \mathcal{I}_{2,1}^F + \mathcal{I}_{2,2}^F + \mathcal{I}_{2,3}^F + \mathcal{I}_{2,4}^F, \quad (3.49)$$

up to some scale free terms. To see this, we have

$$\frac{1}{\ell^2} (\mathcal{T}_{21,1}^Q + \mathcal{T}_{22,1}^Q) = \mathcal{I}_{2,1}^F, \quad (3.50)$$

$$\frac{1}{\ell^2} \mathcal{T}_{21,3}^Q = \mathcal{I}_{2,2}^F + \frac{2q \cdot p_{23}}{\ell^2 (2K_{23} \cdot \ell)(2\ell \cdot p_2)p_{23}^2}, \quad K_{23} \equiv (p_{23}^2)q - (2q \cdot p_{23})p_{23}, \quad (3.51)$$

$$\frac{1}{\ell^2} \mathcal{T}_{21,4}^Q = \mathcal{I}_{2,4}^F - \frac{2q \cdot p_{23}}{\ell^2 (2K_{23} \cdot \ell)(p_{23}^2)^2} + \frac{(2q \cdot p_{23})^2}{\ell^2 (2K_{23} \cdot \ell)^2 p_{23}^2} - \frac{(2q \cdot p_{23})(2q \cdot \ell)}{\ell^2 (2K_{23} \cdot \ell)^2 p_{23}^2}, \quad (3.52)$$

and

$$\frac{1}{\ell^2} (\mathcal{T}_{21,2}^Q + \mathcal{T}_{22,2}^Q) = \mathcal{I}_{2,3}^F - \frac{(2q \cdot p_{23})^2}{\ell^2 (2K'_1 \cdot \ell)(2K_{23} \cdot \ell)p_{23}^2}. \quad (3.53)$$

Thus confirming the equivalence.

For tree diagram \mathcal{T}_3^Q , we have two contributing diagrams as shown in Figure 2.c, and we get

$$\begin{aligned}\mathcal{T}_{31}^Q &= \mathcal{A}_4(\hat{1}, 2, \ell, \hat{P}) \frac{1}{2\ell \cdot p_{12} + p_{12}^2} \mathcal{A}_4(-\hat{P}, -\ell, \hat{3}, 4) \\ &= \left(\frac{1}{2\ell \cdot p_2} + \frac{1}{\hat{p}_{12}^2} \right) \frac{1}{-2\ell \cdot p_{34} + p_{34}^2} \left(\frac{1}{-2\ell \cdot \hat{p}_3} + \frac{1}{\hat{p}_{34}^2} \right) \Big|_{z_{12}^+} \equiv \mathcal{T}_{31,1}^Q + \mathcal{T}_{31,2}^Q + \mathcal{T}_{31,3}^Q + \mathcal{T}_{31,4}^Q, \quad (3.54)\end{aligned}$$

as well as

$$\begin{aligned}\mathcal{T}_{32}^{\mathcal{Q}} &= \mathcal{A}_3(\widehat{P}, 4, P') \frac{1}{2\ell \cdot \widehat{p}_{12} + \widehat{p}_{12}^2} \mathcal{A}_4(-P', \widehat{1}, 2, \alpha\ell) \frac{1}{-2\ell \cdot p_3} \mathcal{A}_3(-\widehat{P}, -\ell, \widehat{3}) \\ &= \frac{2q \cdot \ell}{(2\ell \cdot p_2)(2\ell \cdot p_3)(2K_3 \cdot \ell)} + \frac{1}{-2\ell \cdot \widehat{p}_{34} + \widehat{p}_{34}^2} \left(\frac{1}{2\ell \cdot p_2} + \frac{1}{\widehat{p}_{12}^2} \right) \frac{1}{-2\ell \cdot p_3} \Big|_{z_3},\end{aligned}\quad (3.55)$$

where $K_3 \equiv (p_{34}^2)q - (2q \cdot p_{34})p_3$,

$$\alpha_{34} = \frac{\widehat{p}_{34}^2}{2\ell \cdot \widehat{p}_{34}} \Big|_{z=z_3}.$$

Again the first term in (3.55) is scale-free, while the second and third term are denoted as $\mathcal{T}_{32,1}^{\mathcal{Q}}, \mathcal{T}_{32,2}^{\mathcal{Q}}$. The result $\frac{1}{\ell^2} \mathcal{T}_3^{\mathcal{Q}}$ is equivalent to

$$\frac{1}{\ell^2} \left(\frac{1}{-2\ell \cdot p_3} + \frac{1}{p_{34}^2} \right) \frac{1}{-2\ell \cdot p_{34} + p_{34}^2} \left(\frac{1}{2\ell \cdot p_2} + \frac{1}{p_{12}^2} \right) \equiv \mathcal{I}_{3,1}^{\mathcal{F}} + \mathcal{I}_{3,2}^{\mathcal{F}} + \mathcal{I}_{3,3}^{\mathcal{F}} + \mathcal{I}_{3,4}^{\mathcal{F}}, \quad (3.56)$$

up to some scale free terms, which can be confirmed by

$$\frac{1}{\ell^2} (\mathcal{T}_{31,1}^{\mathcal{Q}} + \mathcal{T}_{32,1}^{\mathcal{Q}}) = \mathcal{I}_{3,1}^{\mathcal{F}}, \quad (3.57)$$

$$\frac{1}{\ell^2} \mathcal{T}_{31,2}^{\mathcal{Q}} = \mathcal{I}_{3,3}^{\mathcal{F}} - \frac{2q \cdot p_{34}}{\ell^2 (2K_{34} \cdot \ell) (2\ell \cdot p_2) p_{34}^2}, \quad K_{34} \equiv (p_{34}^2)q - (2q \cdot p_{34})p_{34}, \quad (3.58)$$

$$\frac{1}{\ell^2} \mathcal{T}_{31,4}^{\mathcal{Q}} = \mathcal{I}_{3,4}^{\mathcal{F}} - \frac{2q \cdot p_{34}}{\ell^2 (2K_{34} \cdot \ell) (p_{34}^2)^2} + \frac{(2q \cdot p_{34})^2}{\ell^2 (2K_{34} \cdot \ell)^2 p_{34}^2} - \frac{(2q \cdot p_{34})(2q \cdot \ell)}{\ell^2 (2K_{34} \cdot \ell)^2 p_{34}^2}, \quad (3.59)$$

and

$$\frac{1}{\ell^2} (\mathcal{T}_{31,3}^{\mathcal{Q}} + \mathcal{T}_{32,2}^{\mathcal{Q}}) = \mathcal{I}_{3,2}^{\mathcal{F}} - \frac{(2q \cdot p_{34})^2}{\ell^2 (2K_3 \cdot \ell) (2K_{34} \cdot \ell) p_{34}^2}. \quad (3.60)$$

For tree diagram $\mathcal{T}_4^{\mathcal{Q}}$, we have two contributing diagrams as shown in Figure 2.d, and we get

$$\begin{aligned}\mathcal{T}_{41}^{\mathcal{Q}} &= \mathcal{A}_4(-\ell, 4, \widehat{1}, \widehat{P}) \frac{1}{-2\ell \cdot p_{41} + p_{41}^2} \mathcal{A}_4(-\widehat{P}, 2, \widehat{3}, \ell) \\ &= \left(\frac{1}{-2\ell \cdot p_4} + \frac{1}{\widehat{p}_{41}^2} \right) \frac{1}{-2\ell \cdot p_{41} + p_{41}^2} \left(\frac{1}{2\ell \cdot \widehat{p}_3} + \frac{1}{\widehat{p}_{23}^2} \right) \Big|_{z_{41}} \equiv \mathcal{T}_{41,1}^{\mathcal{Q}} + \mathcal{T}_{41,2}^{\mathcal{Q}} + \mathcal{T}_{41,3}^{\mathcal{Q}} + \mathcal{T}_{41,4}^{\mathcal{Q}},\end{aligned}\quad (3.61)$$

as well as

$$\begin{aligned}\mathcal{T}_{42}^{\mathcal{Q}} &= \mathcal{A}_4(-\alpha\ell, 4, \widehat{1}, P') \frac{1}{-2\ell \cdot \widehat{p}_{41} + \widehat{p}_{41}^2} \mathcal{A}_3(-P', 2, \widehat{P}) \frac{1}{2\ell \cdot p_3} \mathcal{A}_3(-\widehat{P}, \widehat{3}, \ell) \\ &= \frac{2q \cdot \ell}{(2\ell \cdot p_4)(2\ell \cdot p_3)(2K'_3 \cdot \ell)} + \left(\frac{1}{-2\ell \cdot p_4} + \frac{1}{\widehat{p}_{41}^2} \right) \frac{1}{-2\ell \cdot \widehat{p}_{41} + \widehat{p}_{41}^2} \frac{1}{2\ell \cdot p_3} \Big|_{z_3},\end{aligned}\quad (3.62)$$

where $K'_3 \equiv (p_{41}^2)q + (2q \cdot p_{41})p_3$,

$$\alpha_{34} = \frac{\widehat{p}_{41}^2}{2\ell \cdot \widehat{p}_{41}} \Big|_{z=z_3}.$$

The first term in (3.62) is scale free, while the second and third terms are denoted as $\mathcal{T}_{42,1}^{\mathcal{Q}}, \mathcal{T}_{42,2}^{\mathcal{Q}}$. The result $\frac{1}{\ell^2} \mathcal{T}_4^{\mathcal{Q}}$ is equivalent to

$$\frac{1}{\ell^2} \left(\frac{1}{-2\ell \cdot p_4} + \frac{1}{p_{41}^2} \right) \frac{1}{-2\ell \cdot p_{41} + p_{41}^2} \left(\frac{1}{2\ell \cdot p_3} + \frac{1}{p_{23}^2} \right) \equiv \mathcal{I}_{4,1}^{\mathcal{F}} + \mathcal{I}_{4,2}^{\mathcal{F}} + \mathcal{I}_{4,3}^{\mathcal{F}} + \mathcal{I}_{4,4}^{\mathcal{F}}, \quad (3.63)$$

up to some scale free terms, which can be confirmed by

$$\frac{1}{\ell^2} (\mathcal{T}_{41,1}^{\mathcal{Q}} + \mathcal{T}_{42,1}^{\mathcal{Q}}) = \mathcal{I}_{4,1}^{\mathcal{F}}, \quad (3.64)$$

$$\frac{1}{\ell^2} \mathcal{T}_{41,2}^{\mathcal{Q}} = \mathcal{I}_{4,2}^{\mathcal{F}} + \frac{2q \cdot p_{41}}{\ell^2 (2K_{41} \cdot \ell) (2\ell \cdot p_4) p_{41}^2}, \quad K_{41} \equiv (p_{41}^2)q - (2q \cdot p_{41})p_{41}, \quad (3.65)$$

$$\frac{1}{\ell^2} \mathcal{T}_{41,4}^{\mathcal{Q}} = \mathcal{I}_{4,4}^{\mathcal{F}} - \frac{2q \cdot p_{41}}{\ell^2 (2K_{41} \cdot \ell) (p_{41}^2)^2} + \frac{(2q \cdot p_{41})^2}{\ell^2 (2K_{41} \cdot \ell)^2 p_{41}^2} - \frac{(2q \cdot p_{41})(2q \cdot \ell)}{\ell^2 (2K_{41} \cdot \ell)^2 p_{41}^2}, \quad (3.66)$$

and

$$\frac{1}{\ell^2} (\mathcal{T}_{41,3}^{\mathcal{Q}} + \mathcal{T}_{42,2}^{\mathcal{Q}}) = \mathcal{I}_{4,3}^{\mathcal{F}} - \frac{(2q \cdot p_{41})^2}{\ell^2 (2K'_3 \cdot \ell) (2K_{41} \cdot \ell) p_{41}^2}. \quad (3.67)$$

The above detailed computations shows that, the result of recursive formula (2.33) is equivalent to the one of Feynman diagram method up to some scale free terms.

3.3 The one-loop four-point amplitude in Yang-Mills theory

Now let us take a quick glance on the well studied example, the one-loop four-gluon all plus helicity amplitude $A^{1\text{-loop}}(1^+, 2^+, 3^+, 4^+)$ in planar Yang-Mills theory. The integrand of the original \mathcal{Q} -cut representation, after dropping some scale free terms, takes [1]

$$\mathcal{I}^{\mathcal{Q}} \sim \frac{[1\ 2][3\ 4]}{\langle 1\ 2 \rangle \langle 3\ 4 \rangle} \frac{(\mu^2 - \ell^2)^2}{\ell^2 (2\ell \cdot p_1) (p_{12}^2 - 2\ell \cdot p_{12}) (-2\ell \cdot p_4)} + \text{Cyclic}\{1, 2, 3, 4\}. \quad (3.68)$$

From the perspective of recursive formula (2.33), the tree diagram $\mathcal{T}^{\mathcal{Q}}$ is a sum over four tree diagrams, denoted as

$$\begin{aligned} \mathcal{T}^{\mathcal{Q}} = & \mathcal{T}_1^{\mathcal{Q}}(\ell, -\ell, 1^+, 2^+, 3^+, 4^+) + \mathcal{T}_2^{\mathcal{Q}}(\ell, -\ell, 2^+, 3^+, 4^+, 1^+) \\ & + \mathcal{T}_3^{\mathcal{Q}}(\ell, -\ell, 3^+, 4^+, 1^+, 2^+) + \mathcal{T}_4^{\mathcal{Q}}(\ell, -\ell, 4^+, 1^+, 2^+, 3^+). \end{aligned} \quad (3.69)$$

To compute $\mathcal{T}_i^{\mathcal{Q}}$'s, we should choose an appropriate momentum deformation. Different momentum deformation leads to different factorization of these tree amplitudes. Although the final result will be the same, the intermediate terms will be quite different. We can choose a deformation such that the computation is as simple as possible. Furthermore, the four $\mathcal{T}_i^{\mathcal{Q}}$'s are in fact independent, so each $\mathcal{T}_i^{\mathcal{Q}}$ could have its own momentum deformation, which makes the computation more flexible. In the following computations, we will take advantage of this freedom.

Let us now take $\mathcal{T}_1^{\mathcal{Q}}(\ell, -\ell, 1^+, 2^+, 3^+, 4^+)$ as example, and assume the internal loop to be massive scalar for simplicity⁶. Let us choose the following momentum deformation,

$$|\widehat{2}\rangle = |2\rangle - z|3\rangle \quad , \quad |\widehat{3}\rangle = |3\rangle + z|2\rangle \quad . \quad (3.70)$$

Since by definition the one-loop integrand $\mathcal{I}_3^{\mathcal{Q}} = 0$, we get only one $\mathcal{R}'_{B,2}$ -type contribution,

$$\mathcal{T}_1^{\mathcal{Q}} = \sum_h A(-\ell^s, 1^+, \widehat{2}^+, \widehat{P}^h) \frac{1}{p_{12}^2 - 2\ell \cdot p_{12}} A(-\widehat{P}^{-h}, \widehat{3}^+, 4^+, \ell^s) \quad , \quad (3.71)$$

where the helicity sum is over all possible states $(+, -, s)$. From the results of tree-level amplitudes in [2], we get the non-vanishing contribution

$$A(-\ell^s, 1^+, \widehat{2}^+, \widehat{P}^s) \frac{1}{p_{12}^2 - 2\ell \cdot p_{12}} A(-\widehat{P}^s, \widehat{3}^+, 4^+, \ell^s) = \frac{[2\ 1]}{\langle 1\ \widehat{2} \rangle} \frac{\mu^2 - \ell^2}{\langle 1| - \ell|1 \rangle} \frac{1}{p_{12}^2 - 2\ell \cdot p_{12}} \frac{[4\ \widehat{3}]}{\langle 3\ 4 \rangle} \frac{\mu^2 - \ell^2}{\langle 4|\ell|4 \rangle} \quad , \quad (3.72)$$

where $z = \frac{p_{12}^2 - 2\ell \cdot p_{12}}{2q \cdot (p_{12} - \ell)}$. Using the momentum conservation identity

$$p_1 + \widehat{p}_2 = -(\widehat{p}_3 + p_4) \rightarrow (p_1 + \widehat{p}_2)^2 = (\widehat{p}_3 + p_4)^2 \rightarrow \frac{[4\ \widehat{3}]}{\langle 1\ \widehat{2} \rangle} = \frac{[2\ 1]}{\langle 3\ 4 \rangle} = \frac{[4\ 3]}{\langle 1\ 2 \rangle} \quad , \quad (3.73)$$

we instantly get

$$\mathcal{T}_1^{\mathcal{Q}} = \frac{[1\ 2][3\ 4]}{\langle 1\ 2 \rangle \langle 3\ 4 \rangle} \frac{(\mu^2 - \ell^2)^2}{(-2\ell \cdot p_1)(p_{12}^2 - 2\ell \cdot p_{12})(2\ell \cdot p_4)} \quad . \quad (3.74)$$

So $\frac{1}{\ell^2} \mathcal{T}_1^{\mathcal{Q}}$ equals to a term in (3.69). Similarly, the BCFW deformation

$$\mathcal{T}_2^{\mathcal{Q}}(\ell, -\ell, 2^+, \widehat{3}^+, \widehat{4}^+, 1^+) \quad , \quad \mathcal{T}_3^{\mathcal{Q}}(\ell, -\ell, 3^+, \widehat{4}^+, \widehat{1}^+, 2^+) \quad , \quad \mathcal{T}_4^{\mathcal{Q}}(\ell, -\ell, 4^+, \widehat{1}^+, \widehat{2}^+, 3^+)$$

will produce the other three terms respectively. This simple example is illustrative to show how the terms computed by recursive formula (2.33) are corresponding to the terms computed by the original \mathcal{Q} -cut representation.

4 Conclusion

In this note, we have taken initial steps for constructing one-loop integrand by combining the BCFW deformation and the \mathcal{Q} -cut construction. We have obtained a recursive formula (2.33), where the one-loop integrand is given by one-loop integrands with lower number of external legs, and tree-level amplitudes. We have presented explicit examples to show the equivalence of our result with the one given by Feynman diagrams and \mathcal{Q} -cut representation, up to scale free terms.

There are several possible applications of the recursive formula (2.33). The first one is to consider the one-loop factorization limit $A_L^{\text{tree}} A_R^{1\text{-loop}} + A_L^{1\text{-loop}} A_R^{\text{tree}} + A_L^{\text{tree}} \mathcal{S} A_R^{\text{tree}}$. It is easy to see that, in the recursive

⁶It is massive in 4-dim, but null in higher dimension.

formula, \mathcal{R}_A^Q contributes to the first two factorization limits, while \mathcal{R}_B^Q contributes to the third term. The \mathcal{R}_B^Q part contains six terms, so naively the kernel \mathcal{S} could be very complicated. However, it could be the case that some terms do not contribute, or their contributions simplify a lot in the factorization limit. It would be interesting to investigate if we can find some compact form for \mathcal{S} or not. Using the recursive formula, we can also study the behavior of integrands in certain limits, for instance the single/double soft limit and the one-loop split function. It is also possible to study the rational part of one-loop amplitudes when constructed using 4-dimensional unitarity cut method, especially if we could write down some recursive relation for the rational part, based on our formula. Finally, generalizations to higher loops and massive external legs, which are a very important open questions in the original Q -cut representation, deserves to be investigated along this direction as well.

Acknowledgments

BF, RH and ML is supported by Qiu-Shi Funding and the National Natural Science Foundation of China (NSFC) with Grant No.11135006, No.11125523 and No.11575156. RH would also like to acknowledge the supporting from Chinese Postdoctoral Administrative Committee. SH acknowledges support from the Thousand Young Talents program and the Key Research Program of Frontier Sciences of CAS (Grant No. QYZDBSSW-SYS014).

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